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Bi-Harmonic Analysis in a Perforated Strip

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VI. *Bi-Harmonic Analysis in a Perforated Strip.*

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Contents.

§ 1. Introduction.

A method of solving the biharmonic equation in a region bounded externally by two parallel straight lines and internally by a circle was given by one of the authors in a recent paper.* General formulæ were developed, but these were restricted to solutions symmetrical about both co-ordinate axes, and were applied to only one special problem

* 'Phil. Trans.,' A, vol. 229, p. 49 (1930); this paper will be referred to as "A."

of elasticity. In the present paper the analysis is generalized to include unsymmetrical solutions, and the formulæ are developed to a point at which it becomes possible to solve any problem of stress within the specified boundaries. Two important special stress-systems—that corresponding to pure bending-moment, and that giving bending-moment with shear—are worked out in detail. A number of other interesting systems may be discussed by the aid of the results given. In addition, only slight modifications are needed to make the equations applicable to the slow motion of a viscous fluid.

In the general discussion of the problem of stress it is found convenient to distinguish three types of stress-systems: (1) those corresponding to forces transmitted from infinity, (2) those produced by self-equilibrating tractions on the circular hole, (3) those produced by forces and couples acting at the origin in an unperforated strip and corresponding, in the perforated strip, to special distributions of tractions on the hole. The third type of solution differs from the others, the stresses being dependent upon the elastic constants. When the stress-system is symmetrical about both axes, as was supposed in "A," solutions of type (3) are absent. They were, however, considered in an earlier paper.* Systems of types (1) and (2) do not require distinct kinds of analysis, but are separated for a different reason. The three fundamental solutions of type (1), since they correspond to boundaries free from stress, are of the nature of complementary functions. Any multiple of one of them may be added to an existing solution without affecting the boundary conditions; only the infinity conditions are modified. In this way, as will be seen, it is possible to adjust the total stress over any two transverse sections of the strip so that the shear and bending-moment have prescribed values.

§ 2. Preliminary Analysis.

Let the strip be bounded in the x, y — plane by the lines $y = \pm b$, and perforated by a hole of radius a ($< b$), with its centre at the origin. Polar co-ordinates (r, θ) will also be used, these being taken in the way shown in fig. 1, the initial line being OY and the positive direction of θ clockwise.

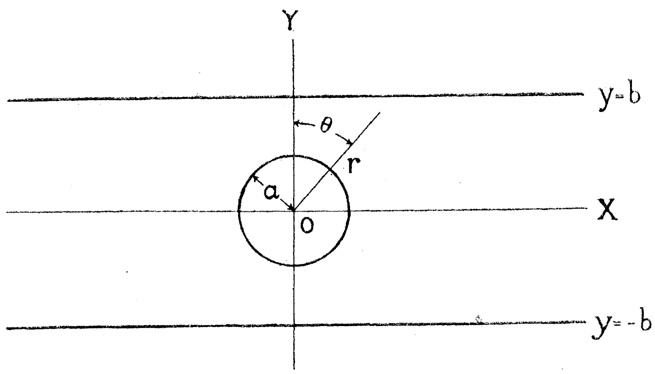

Fig. 1.

* 'Proc. Roy. Soc.' A, vol. 124, p. 89 (1929); this paper will be referred to as "B."

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

This convention was used in the previous paper, and has certain advantages. The relation between the two sets of co-ordinates is

$$\left.\begin{aligned} x &= r \sin \theta \\ y &= r \cos \theta \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots (1)$$

As in the previous paper, we express our formulæ in terms of dimensionless co-ordinates, derived from these by the equations

$$\xi = x/b, \eta = y/b, \rho = r/b, \quad \ldots \ldots \ldots \ldots (2)$$

and use also the abbreviation

$$\lambda = a/b \quad \ldots \ldots \ldots \ldots (3)$$

A system of generalized plane stress may be specified by a stress-function χ satisfying the bi-harmonic equation

$$\nabla^4 \chi \equiv \frac{\partial^4 \chi}{\partial x^4} + 2\frac{\partial^4 \chi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \chi}{\partial y^4} = 0. \quad \ldots \ldots \ldots \ldots (4)$$

The stress-components relative to Cartesian co-ordinates are

$$\widehat{xx} = \frac{1}{b^2}\frac{\partial^2 \chi}{\partial \eta^2}, \quad \widehat{yy} = \frac{1}{b^2}\frac{\partial^2 \chi}{\partial \xi^2}, \quad \widehat{xy} = -\frac{1}{b^2}\frac{\partial^2 \chi}{\partial \xi\, \partial \eta}. \quad \ldots \ldots \ldots (5)$$

Those relative to the polar co-ordinates are

$$\left.\begin{aligned} \widehat{rr} &= \frac{1}{b^2}\left[\frac{1}{\rho^2}\frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial \chi}{\partial \rho}\right] \\ \widehat{\theta\theta} &= \frac{1}{b^2}\frac{\partial^2 \chi}{\partial \rho^2}, \quad \widehat{r\theta} = -\frac{1}{b^2}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \chi}{\partial \theta}\right) \end{aligned}\right\} \quad \ldots \ldots \ldots (6)$$

The general problem is that of finding a solution of (4) corresponding to definite values of $\widehat{rr}$ and $\widehat{r\theta}$ on the circle $\rho = \lambda$, to zero values of $\widehat{yy}$ and $\widehat{xy}$ on the lines $y = \pm b$ and to certain conditions, to be specified later, at infinity. The condition of zero tractions on the straight edges is not essential. The slight modifications needed in order to account for tractions on the edges will be indicated shortly.*

As a first step in the solution, we write

$$\chi = \chi' + \chi'' + \chi''' \quad \ldots \ldots \ldots \ldots (7)$$

where χ' is a stress-function giving the required stresses at infinity, while χ'' accounts for the unbalanced part of the tractions on the hole. Both χ' and χ'' are to give zero tractions on the straight edges. χ''' will thus correspond to zero tractions at infinity,

* See footnote p. 175.

zero tractions on the straight edges and self-equilibrating tractions on the circular boundary.

The infinity conditions will be specified by giving the resultant tension, bending-moment and shear across a distant section of the strip. χ' may then be built up from three well-known types of stress-function. These are

$$(a) \quad \chi = \tfrac{1}{2}b^2\mathrm{T}\eta^2\,;$$

this gives $\widehat{xx} = \mathrm{T}$, $\widehat{yy} = \widehat{xy} = 0$, *i.e.*, a uniform tension.

$$(b) \quad \chi = \tfrac{1}{4}\mathrm{M}b\eta^2.$$

The stresses are $\widehat{xx} = \dfrac{3}{2}\dfrac{\mathrm{M}}{b}\eta$, $\widehat{yy} = \widehat{xy} = 0$, giving a bending-moment M across every section.

$$(c) \quad \chi = \tfrac{1}{4}\,\mathrm{P}b\,\xi\,\eta\,(\eta^2 - 3).$$

The stresses now are $\widehat{xx} = \dfrac{3}{2}\dfrac{\mathrm{P}}{b}\xi\,\eta$, $\widehat{yy} = 0$, $\widehat{xy} = \dfrac{3\mathrm{P}}{4b}(\eta^2 - 1)$, giving a total shear P across every section and a bending-moment Px tending to infinity with x.

Stresses at infinity are also introduced automatically with χ'', and these must be taken into account in the choice of χ'. The infinity stresses due to χ'' may, however, be predicted in advance, as indicated in what follows.

In determining χ'', the first step is to find the resultant of the tractions on the circular boundary. This resultant may be represented as a force and a couple acting at the origin. The stress functions corresponding to longitudinal and transverse forces acting in an unperforated strip have been given in a previous paper* in the form of integrals, expansions of which will be given in the next section. The solution for a couple will also be given, and χ'' will be taken as a linear combination of these three solutions.

Neither χ' nor χ'' will satisfy conditions of zero traction on the circular boundary. It will be necessary to determine the tractions due to them, and to subtract these from the given tractions. The differences thus obtained will be the tractions that must be produced by χ''', and it is to be observed that they form a self-equilibrating system. χ''' will therefore be independent of the elastic constants. It will be determined by the methods of the previous paper. To clarify the analysis, the solution will be presented in four parts, χ''' being regarded as the sum of four stress-functions, (i) even in both x and y, (ii) even in x and odd in y, (iii) odd in x and even in y, (iv) odd in both x and y. Of these, (i) may be found from formulæ proved in the previous paper. These formulæ will be quoted without proof, but, in order to simplify the new results that follow, a slight change of notation will be made. Formulæ for (iii) were published without proof in a recent communication†; the proofs, and those of the new results for (ii) and (iv), will now be given.

* "B," pp. 97–105; see also pp. 106 and 112.

† 'Proc. Third Int. Cong. App. Mechanics,' Stockholm, vol. 2, p. 74 (1930).

§ 3. Expansions of Stress-Functions Representing Forces Located at the Origin.

The stress-function representing a force P acting at the origin in the direction OX when the strip is unperforated may be written*

$$\chi_a = \frac{Pb}{4\pi(1-\sigma)}\left[(1-2\sigma)\,\xi\log\rho + 2(1-\sigma)\,\eta\theta + 2\,\Phi_a\right] \quad . \; . \; . \; . \; . \quad (8)$$

where

$$\left.\begin{aligned}\Phi_a &= \int_0^\infty \frac{u\eta s\mathrm{S} - (s+uc)\,\mathrm{C}}{u^2\Sigma}\{u-(1-2\sigma)\}\,e^{-u}\sin u\,\xi\,du \\ &+ \int_0^\infty \frac{s\,\mathrm{C} - \eta c\,\mathrm{S}}{u\Sigma}\{2(1-\sigma)-u\}\,e^{-u}\sin u\xi\,du\end{aligned}\right\}, \quad . \; . \; . \; . \quad (9)$$

σ being the modified value of Poisson's Ratio† occurring in the theory of "generalized plane stress," and the other new symbols being defined as follows :—

$$\left.\begin{aligned} s = \sinh u,\ c = \cosh u,\ \mathrm{S} = \sinh \eta u, \\ \mathrm{C} = \cosh \eta u,\ \Sigma = \sinh 2u + 2u.\end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (10)$$

To expand Φ_a in a convenient form we first reduce it to

$$\begin{aligned}\Phi_a &= \tfrac{1}{2}\int_0^\infty \frac{\eta\mathrm{S}\sin u\,\xi}{u\,\Sigma}\{2u - 3 + 4\sigma - e^{-2u}\}du \\ &+ \tfrac{1}{2}\int_0^\infty \frac{\mathrm{C}\sin u\,\xi}{u^2\Sigma}\{(1-2\sigma) + 2(1-2\sigma)\,u - 2u^2 - (1-2\sigma)\,e^{-2u}\}du.\end{aligned}$$

In this we substitute the expansions

$$\left.\begin{aligned}\eta\,\mathrm{S}\sin u\,\xi &= \tfrac{1}{2}\left[\frac{u^2\rho^3}{2}\sin\theta + \sum_{n=1}^{\infty}\left\{\frac{u^{2n}\rho^{2n+1}}{(2n)!} + \frac{u^{2n+2}\rho^{2n+3}}{(2n+2)!}\right\}\sin(2n+1)\,\theta\right] \\ \mathrm{C}\sin u\,\xi &= \sum_{n=0}^{\infty}\frac{(u\rho)^{2n+1}}{(2n+1)!}\sin(2n+1)\,\theta\end{aligned}\right\}, \quad (11)$$

of which the second is well known, while the first is readily derivable from a well-known result ("A," p. 58). Then, omitting the term in $\rho\sin\theta$, which is trivial, as it contributes nothing to the stresses, we obtain

$$\Phi_a = {}^a b_0\,\rho^3\sin\theta + \sum_{n=1}^{\infty}({}^a a_n + {}^a b_n\,\rho^2)\,\rho^{2n+1}\sin(2n+1)\,\theta, \quad . \; . \; . \; . \; . \quad (12)$$

* "B," pp. 97–99 and p. 106.

† Filon, 'Phil. Trans.' A, vol. 201, p. 67 (1903); Love, "Theory of Elasticity," 4th edn., p. 138, Camb. (1927).

where

$$^{a}a_n = \frac{1}{4\,.\,(2n+1)\,!}\{2\,(2n+3-4\sigma)\,I_{2n} - (6n+1-8n\,\sigma)\,I_{2n-1} - 4\,I_{2n+1} - (2n+3-4\sigma)\,J_{2n-1}\}, \quad . \quad . \quad . \quad . \quad (13)$$

$$^{a}b_n = \frac{1}{4\,.\,(2n+2)\,!}\{2\,I_{2n+2} - (3-4\sigma)\,I_{2n+1} + J_{2n+1}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and I_s, J_s denote the integrals ("A," p. 59)

$$\left.\begin{aligned} I_s &= \int_0^\infty \frac{u^s}{\Sigma}\,du \\ J_s &= \int_0^\infty \frac{u^s}{\Sigma}\,e^{-2u}\,du \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

For the stress-function we now have

$$\chi_a = \frac{Pb}{4\pi\,(1-\sigma)}\left[(1-2\sigma)\,\xi\log\rho + 2\,(1-\sigma)\,\eta\,\theta\right] + Pb\left[{}^{a}b'_0\,\rho^3\sin\theta + \sum_{n=1}^{\infty}({}^{a}a'_n + {}^{a}b'_n\,\rho^2)\,\rho^{2n+1}\sin(2n+1)\,\theta\right] \quad . \quad . \quad (16)$$

where

$$^{a}a'_n = \frac{^{a}a_n}{2\pi\,(1-\sigma)}\quad,\quad {}^{a}b'_n = \frac{^{a}b_n}{2\pi\,(1-\sigma)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The corresponding stresses are

$$\left.\begin{aligned} \widehat{rr} &= -\frac{P}{4\pi b}\,\frac{3-2\sigma}{1-\sigma}\,\frac{\sin\theta}{\rho} \\ &\quad -\frac{P}{b}\sum_{n=0}^{\infty}\{2n\,(2n+1)\,{}^{a}a'_n + (2n-1)\,(2n+2)\,{}^{a}b'_n\rho^2\}\,\rho^{2n-1}\sin(2n+1)\,\theta \\ \widehat{r\theta} &= -\frac{P}{4\pi b}\,\frac{1-2\sigma}{1-\sigma}\,\frac{\cos\theta}{\rho} \\ &\quad -\frac{P}{b}\sum_{n=0}^{\infty}\{2n\,(2n+1)\,{}^{a}a'_n + (2n+1)\,(2n+2)\,{}^{a}b'_n\,\rho^2\}\,\rho^{2n-1}\cos(2n+1)\,\theta \\ \widehat{\theta\theta} &= \frac{P}{4\pi b}\,\frac{1-2\sigma}{1-\sigma}\,\frac{\sin\theta}{\rho} \\ &\quad +\frac{P}{b}\sum_{n=0}^{\infty}\{2n\,(2n+1)\,{}^{a}a'_n + (2n+2)\,(2n+3)\,{}^{a}b'_n\,\rho^2\}\,\rho^{2n-1}\sin(2n+1)\,\theta \end{aligned}\right\}. \quad (18)$$

Putting $\rho = \lambda$, we get for the tractions at the rim of the hole

$$\left.\begin{aligned} \widehat{rr} &= \frac{P}{b}\sum_{n=0}^{\infty}{}^{a}p_n\sin(2n+1)\,\theta \\ \widehat{r\theta} &= \frac{P}{b}\sum_{n=0}^{\infty}{}^{a}r_n\cos(2n+1)\,\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where

$$\left.\begin{aligned} {}^{\alpha}p_o &= -\frac{3-2\sigma}{4\pi(1-\sigma)\lambda} + 2\,{}^{\alpha}b'_o\,\lambda, \\ {}^{\alpha}p_n &= -\{2n(2n+1)\,{}^{\alpha}a'_n + (2n-1)(2n+2)\,{}^{\alpha}b'_n\,\lambda^2\}\,\lambda^{2n-1},\ n>0 \\ {}^{\alpha}r_o &= -\frac{1-2\sigma}{4\pi(1-\sigma)\lambda} - 2\,{}^{\alpha}b'_o\,\lambda, \\ {}^{\alpha}r_n &= -\{2n(2n+1)\,{}^{\alpha}a'_n + (2n+1)(2n+2)\,{}^{\alpha}b'_n\,\lambda^2\}\,\lambda^{2n-1},\ n>0 \end{aligned}\right\} . \qquad (20)$$

Numerical values for these coefficients will be given later.

The stress-function for a force P acting at the origin in the direction OY is ("B," *loc. cit.*)

$$\chi_\beta = \frac{Pb}{4\pi(1-\sigma)}\left[(1-2\sigma)\,\eta\log\rho - 2(1-\sigma)\,\xi\theta + 2\,\Phi_\beta\right] \quad . \quad . \quad . \qquad (21)$$

where

$$\left.\begin{aligned} \Phi_\beta = &\int_0^\infty \left\{\frac{u\eta c\mathrm{C} - (c+us)\,\mathrm{S}}{u^2\,\Sigma'}\cos u\,\xi - \frac{\eta(\eta^2-3)}{4u^2}\right\}\{2(1-\sigma)+u\}e^{-u}du \\ &+ \int_0^\infty \left\{\frac{\eta s\mathrm{C} - c\mathrm{S}}{u\Sigma'}\cos u\xi - \frac{\eta(\eta^2-1)}{4u}\right\}\{(1-2\sigma)+u\}\,e^{-u}\,du, \end{aligned}\right\} \qquad (22)$$

where

$$\Sigma' = \sinh 2u - 2u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

The convergence of the integrals in (22) has been secured in the way described in the previous paper ("B," pp. 174, 112). Proceeding as before, we obtain, after some reduction,

$$\begin{aligned} \Phi_\beta = &\tfrac{1}{2}\int_0^\infty \frac{\eta\mathrm{C}\cos u\xi}{u\Sigma'}\{2u + (3-4\sigma) + e^{-2u}\}\,du \\ &- \tfrac{1}{2}\int_0^\infty \frac{\mathrm{S}\cos u\xi}{u^2\Sigma'}\{2u^2 + 4(1-\sigma)\,u + 2(1-\sigma) + 2(1-\sigma)\,e^{-2u}\}\,du \\ &- \frac{\eta}{4}\int_0^\infty \left\{(\eta^2-1) + \frac{2(1-\sigma)\,\eta^2 - 2(2-\sigma)}{u} + \frac{2(1-\sigma)(\eta^2-3)}{u^2}\right\}e^{-u}\,du. \end{aligned}$$

In the first two integrals we now use the expansions

$$\left.\begin{aligned} \mathrm{S}\cos u\,\xi &= \sum_{n=0}^{\infty}\frac{(u\rho)^{2n+1}}{(2n+1)!}\cos(2n+1)\,\theta \\ \eta\mathrm{C}\cos u\xi &= \tfrac{1}{2}\Big[2\rho\cos\theta + \tfrac{1}{2}u^2\rho^3\cos\theta \\ &\quad + \sum_{n=1}^{\infty}\left\{\frac{u^{2n}\rho^{2n+1}}{(2n)!} + \frac{u^{2n+2}\rho^{2n+3}}{(2n+2)!}\right\}\cos(2n+1)\,\theta\Big] \end{aligned}\right\} . \qquad (24)$$

while in the third integral η and η^3 are replaced by $\rho\cos\theta$ and $\tfrac{1}{4}\rho^3(3\cos\theta + \cos 3\theta)$ respectively. Φ_β may then be reduced to the form

$$\Phi_\beta = \sum_{n=0}^{\infty}({}^{\beta}a_n + {}^{\beta}b_n\rho^2)\,\rho^{2n+1}\cos(2n+1)\,\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

where, if $n > 1$, the coefficients are

$$\left.\begin{aligned} {}^{\beta}a_n &= \frac{1}{4\,.\,(2n+1)\,!}\left[2\,(2n-3+4\sigma)\,\mathrm{I}'_{2n} + (6n-1-8n\sigma)\,\mathrm{I}'_{2n-1}\right. \\ &\qquad\qquad \left. - 4\mathrm{I}'_{2n+1} + (2n-3+4\sigma)\,\mathrm{J}'_{2n-1}\right] \\ {}^{\beta}b_n &= \frac{1}{4\,.\,(2n+2)\,!}\left[2\mathrm{I}'_{2n+2} + (3-4\sigma)\,\mathrm{I}'_{2n+1} + \mathrm{J}'_{2n+1}\right] \end{aligned}\right\} \quad (26)$$

the new symbols denoting the integrals

$$\left.\begin{aligned} \mathrm{I}_s' &= \int_0^\infty \frac{u^s}{\Sigma'}\,du \\ \mathrm{J}_s' &= \int_0^\infty \frac{u^s\,e^{-2u}}{\Sigma'}\,du \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (27)$$

These integrals are divergent if $s < 3$, so that the formula for ${}^{\beta}a_1$ cannot be written in the form (26); n may, however, be put equal to 1 in the formula for ${}^{\beta}b_n$. The values of ${}^{\beta}a_0$, ${}^{\beta}a_1$ and ${}^{\beta}b_0$ will now be considered separately; it will be found that convergence is secured through the additional terms coming from the third integral in Φ_β.

We have first

$$\begin{aligned} {}^{\beta}a_0 = \tfrac{1}{2}\int_0^\infty \Big[& \frac{1}{u\Sigma'}\{(1-2\sigma) - 2\,(1-2\sigma)\,u - 2u^2 - (1-2\sigma)\,e^{-2u}\} \\ & + \frac{1}{2u^2}\{u^2 + 2\,(2-\sigma)\,u + 6\,(1-\sigma)\}\,e^{-u}\Big]\,du. \end{aligned}$$

The integrand is

$$\begin{aligned} & \frac{3}{4u^4}\{1 + \mathrm{O}\,(u^2)\}\,\{(1-2\sigma) - 2\,(1-2\sigma)\,u - 2u^2 \\ & \qquad - (1-2\sigma)\,[1 - 2u + 2u^2 - \tfrac{4}{3}u^3 + \mathrm{O}\,(u^4)]\} \\ & + \frac{1}{2u^2}\{u^2 + 2\,(2-\sigma)\,u + 6\,(1-\sigma)\}\,\{1 - u + \tfrac{1}{2}u^2 + \mathrm{O}\,(u^3)\} \\ & = \frac{3}{4u^4}\{-4\,(1-\sigma)\,u^2 + \tfrac{4}{3}\,(1-2\sigma)\,u^3 + \mathrm{O}\,(u^4)\} \\ & \qquad + \frac{1}{2u^2}\{6\,(1-\sigma) - 2\,(1-2\sigma)\,u + \mathrm{O}\,(u^2)\} \\ & = \mathrm{O}\,(1). \end{aligned}$$

Hence the integral converges at the lower limit; it also converges at the upper limit, so that ${}^{\beta}a_0$ is finite. As this term contributes nothing to the stresses, it is unnecessary to consider it further.

The next coefficient to be dealt with is ${}^{\beta}b_0$. We have

$$ {}^{\beta}b_0 = \tfrac{1}{8}\int_0^{\infty}\left[\frac{1}{\Sigma'}\{2u^2+(3-4\sigma)u+ue^{-2u}-\frac{3}{2u^2}\{u^2+2(1-\sigma)u+2(1-\sigma)\}e^{-u}\right]du. $$

It has already been observed that the definitions of I'_s, J'_s in (27) are not valid if $s < 3$; but modifications of the integrals may be used. These will be defined as follows:—

$$ \left.\begin{aligned} I'_0 &= \int_0^{\infty}\left\{\frac{1}{\Sigma'}-\frac{3\,(10+10u+3u^2)}{40u^3}e^{-u}\right\}du \\ I'_1 &= \int_0^{\infty}\left\{\frac{u}{\Sigma'}-\frac{3\,(1+u)}{4u^2}e^{-u}\right\}du \\ I'_2 &= \int_0^{\infty}\left\{\frac{u^2}{\Sigma'}-\frac{3}{4u}e^{-u}\right\}du \\ J'_1 &= \int_0^{\infty}\left\{\frac{u}{\Sigma'}e^{-2u}-\frac{3\,(1-u)}{4u^2}e^{-u}\right\}du \\ J'_2 &= \int_0^{\infty}\left\{\frac{u^2}{\Sigma'}e^{-2u}-\frac{3}{4u}e^{-u}\right\}du \end{aligned}\right\} \quad \ldots\ldots \quad (28) $$

These integrals are all convergent at both limits. In terms of them ${}^{\beta}b_0$ may be written

$$ \begin{aligned} {}^{\beta}b_0 &= \tfrac{1}{8}\,[(3-4\sigma)\,I'_1+2I'_2+J'_1]-\tfrac{3}{16}\int_0^{\infty}e^{-u}\,du \\ &= \tfrac{1}{8}\,[(3-4\sigma)\,I'_1+2I'_2+J'_1-\tfrac{3}{2}]. \quad \ldots\ldots\ldots\ldots \quad (29) \end{aligned} $$

In the same way

$$ \begin{aligned} {}^{\beta}a_1 &= \tfrac{1}{48}\int_0^{\infty}\left[\frac{-8u^3-4(1-4\sigma)u^2+2(5-8\sigma)u-2(1-4\sigma)ue^{-2u}}{\Sigma'}\right. \\ &\qquad\qquad\left.-\frac{3u^2+6(1-\sigma)(1+u)}{u^2}e^{-u}\right]du \\ &= \tfrac{1}{48}\,[-8I'_3-4(1-4\sigma)\,I'_2+2(5-8\sigma)\,I'_1-2(1-4\sigma)\,J'_1-3]. \quad \ldots \quad (30) \end{aligned} $$

The complete stress-function is now

$$ \chi_{\beta} = \frac{Pb}{4\pi(1-\sigma)}\{(1-2\sigma)\,\eta\log\rho+2(1-\sigma)\,\xi\theta\} $$

$$ +Pb\sum_{n=0}^{\infty}({}^{\beta}a'_n+{}^{\beta}b'_n\rho^2)\,\rho^{2n+1}\cos(2n+1)\,\theta, \quad \ldots\ldots\ldots \quad (31) $$

where

$$^{\beta}a'_n = \frac{^{\beta}a_n}{2\pi(1-\sigma)}, \quad ^{\beta}b'_n = \frac{^{\beta}b_n}{2\pi(1-\sigma)}. \quad \quad (32)$$

This gives the following values for the stress-components

$$\left.\begin{aligned}
\widehat{rr} &= -\frac{P}{4\pi b}\frac{3-2\sigma}{1-\sigma}\frac{\cos\theta}{\rho} \\
&\quad -\frac{P}{b}\sum_{n=0}^{\infty}\{2n(2n+1)\,^{\beta}a'_n + (2n-1)(2n+2)\,^{\beta}b'_n\rho^2\}\,\rho^{2n-1}\cos(2n+1)\theta \\
\widehat{r\theta} &= \frac{P}{4\pi b}\frac{1-2\sigma}{1-\sigma}\frac{\sin\theta}{\rho} \\
&\quad +\frac{P}{b}\sum_{n=0}^{\infty}\{2n(2n+1)\,^{\beta}a'_n + (2n+1)(2n+2)\,^{\beta}b'_n\rho^2\}\,\rho^{2n-1}\sin(2n+1)\theta \\
\widehat{\theta\theta} &= \frac{P}{4\pi b}\frac{1-2\sigma}{1-\sigma}\frac{\cos\theta}{\rho} \\
&\quad +\frac{P}{b}\sum_{n=0}^{\infty}\{2n(2n+1)\,^{\beta}a'_n + (2n+2)(2n+3)\,^{\beta}b'_n\rho^2\}\,\rho^{2n-1}\cos(2n+1)\theta
\end{aligned}\right\} . \quad (33)$$

At the rim of the hole these become

$$\left.\begin{aligned}
\widehat{rr} &= \frac{P}{b}\sum_{n=0}^{\infty}\,^{\beta}p_n\cos(2n+1)\theta \\
\widehat{r\theta} &= \frac{P}{b}\sum_{n=0}^{\infty}\,^{\beta}r_n\sin(2n+1)\theta
\end{aligned}\right\}, \quad \quad (34)$$

where

$$\left.\begin{aligned}
^{\beta}p_0 &= -\frac{3-2\sigma}{4\pi(1-\sigma)\lambda} + 2\,^{\beta}b'_0\lambda \\
^{\beta}p_n &= -\{2n(2n+1)\,^{\beta}a'_n + (2n-1)(2n+2)\,^{\beta}b'_n\lambda^2\}\,\lambda^{2n-1}, \quad n > 0 \\
^{\beta}r_0 &= \frac{1-2\sigma}{4\pi(1-\sigma)\lambda} + 2\,^{\beta}b'_0\lambda \\
^{\beta}r_n &= \{2n(2n+1)\,^{\beta}a'_n + (2n+1)(2n+2)\,^{\beta}b'_n\lambda^2\}\,\lambda^{2n-1}, \quad n > 0
\end{aligned}\right\} . . (35)$$

No formula for the stress-function corresponding to a couple at the origin was stated in the earlier paper, but such a formula is easily derived. The stress-function for a force P acting at the point $(0, bh)$ and in the direction of OX is*

$$\chi = \chi_0 + \chi_5 + \chi_6 + \chi_7 + \chi_8, \quad \quad (36)$$

* "B," p. 97. The sign of θ has been changed, and a few other obvious modifications made, in accordance with the notation of the present paper.

where

$$\left.\begin{aligned}
\chi_0 &= \frac{\mathrm{P}b}{4\pi}\left[\frac{1-2\sigma}{1-\sigma}\,\xi\log\rho + 2(\eta-h)\,\theta\right] \\
\chi_5 &= \frac{\mathrm{P}b}{4\pi(1-\sigma)}\int_0^\infty \frac{u\eta s\mathrm{S} - (s+uc)\,\mathrm{C}}{u^2\Sigma}(\mathrm{B}_1+\mathrm{B}_2)\sin u\xi\,du \\
\chi_6 &= \frac{\mathrm{P}b}{4\pi(1-\sigma)}\int_0^\infty \frac{s\mathrm{C} - \eta c\mathrm{S}}{u\Sigma}(\mathrm{B}'_1+\mathrm{B}'_2)\sin u\xi\,du \\
\chi_7 &= \frac{\mathrm{P}b}{4\pi(1-\sigma)}\int_0^\infty \frac{u\eta c\mathrm{C} - (c+us)\,\mathrm{S}}{u^2\Sigma'}(\mathrm{B}_1-\mathrm{B}_2)\sin u\xi\,du \\
\chi_8 &= \frac{\mathrm{P}b}{4\pi(1-\sigma)}\int_0^\infty \frac{c\mathrm{S} - \eta s\mathrm{C}}{u\Sigma'}(\mathrm{B}'_1-\mathrm{B}'_2)\sin u\xi\,du
\end{aligned}\right\}, \quad . \quad (37)$$

and

$$\left.\begin{aligned}
\mathrm{B}_1 &= \{u(1-h) - (1-2\sigma)\}\,e^{-u(1-h)} \\
\mathrm{B}_2 &= \{u(1+h) - (1-2\sigma)\}\,e^{-u(1+h)} \\
\mathrm{B}'_1 &= \{2(1-\sigma) - u(1-h)\}\,e^{-u(1-h)} \\
\mathrm{B}'_2 &= \{2(1-\sigma) - u(1+h)\}\,e^{-u(1+h)}
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (38)$$

If χ is differentiated with respect to h and h is put equal to 0 and Pb replaced by M, the resulting function will be the stress-function for a couple M at the origin. The result is easily seen to be

$$\chi_\gamma = \frac{\mathrm{M}}{2\pi}\left[-\theta + \frac{1}{1-\sigma}\{\tfrac{1}{4}\sin 2\theta + \Phi_\gamma\}\right] \quad . \; . \; . \; . \; . \; . \; . \; . \quad (39)$$

where

$$\begin{aligned}
\Phi_\gamma &= \int_0^\infty \frac{u\eta c\mathrm{C} - (c+us)\,\mathrm{S}}{u\Sigma'}\{u - 2(1-\sigma)\}\,e^{-u}\sin u\xi\,du \\
&\quad + \int_0^\infty \frac{c\mathrm{S} - \eta s\mathrm{C}}{\Sigma'}\{(3-2\sigma) - u\}\,e^{-u}\sin u\,\xi\,du \\
&= \tfrac{1}{2}\int_0^\infty \frac{\eta\mathrm{C}\sin u\xi}{\Sigma'}\{2u - (5-4\sigma) + e^{-2u}\}\,du \\
&\quad + \tfrac{1}{2}\int_0^\infty \frac{\mathrm{S}\sin u\xi}{u\Sigma'}\{2(1-\sigma) + 4(1-\sigma)\,u - 2u^2 + 2(1-\sigma)\,e^{-2u}\}\,du.
\end{aligned}$$

In this we make the substitutions

$$\left.\begin{aligned}
\eta\mathrm{C}\sin u\xi &= \tfrac{1}{2}\sum_{n=1}^{\infty}\left\{\frac{u^{2n+1}\rho^{2n+2}}{(2n+1)!} + \frac{u^{2n-1}\rho^{2n}}{(2n-1)!}\right\}\sin 2n\theta \\
\mathrm{S}\sin u\xi &= \sum_{n=1}^{\infty}\frac{(u\rho)^{2n}\sin 2n\theta}{(2n)!}
\end{aligned}\right\} \quad . \; . \; . \; . \quad (40)$$

Then

$$\Phi_\gamma = \sum_{n=1}^{\infty} ({}^\gamma a_n + {}^\gamma b_n \rho^2) \sin 2n\theta \quad \ldots \ldots \ldots \quad (41)$$

where

$$\left.\begin{aligned} {}^\gamma a_n &= \frac{1}{2\,.\,(2n)\,!}\{2(n+2-2\sigma)\,\Gamma'_{2n} - (3+2\sigma-4n\sigma)\,\Gamma'_{2n-1} \\ &\qquad\qquad - 2\Gamma'_{2n+1} + (n+2-2\sigma)\,J'_{2n-1}\} \\ {}^\gamma b_n &= \frac{1}{4\,.\,(2n+1)\,!}\{2\Gamma'_{2n+2} - (5-4\sigma)\,\Gamma'_{2n+1} + J'_{2n+1}\} \end{aligned}\right\}. \quad (42)$$

The formula for ${}^\gamma a_1$ may be included in (42), provided that Γ'_1, Γ'_2 and J'_2 are defined as in (28).

The value of the stress-function may now be written

$$\chi_\gamma = M\left[-\frac{\theta}{2\pi} - \sum_{n=1}^{\infty} ({}^\gamma a'_n + {}^\gamma b'_n \rho^2)\,\rho^{2n} \sin 2n\theta\right] \quad \ldots \ldots \quad (43)$$

where

$$\left.\begin{aligned} {}^\gamma a'_1 &= \frac{{}^\gamma a_1 + \frac{1}{4}}{2\pi\,(1-\sigma)}, \\ {}^\gamma a'_n &= \frac{{}^\gamma a_n}{2\pi\,(1-\sigma)}, \quad {}^\gamma b'_n = \frac{{}^\gamma b_n}{2\pi\,(1-\sigma)} \end{aligned}\right\} \quad \ldots \ldots \quad (44)$$

The corresponding stress-components are

$$\left.\begin{aligned} \widehat{rr} &= -\frac{M}{b^2}\sum_{n=1}^{\infty}\{2n(2n-1)\,{}^\gamma a'_n + (2n-2)(2n+1)\,\rho^2\,{}^\gamma b'_n\}\,\rho^{2n-2}\sin 2n\theta \\ \widehat{r\theta} &= -\frac{M}{b^2}\left[\frac{1}{2\pi\rho^2} + \sum_{n=1}^{\infty}\{2n(2n-1)\,{}^\gamma a'_n + 2n(2n+1)\,\rho^2\,{}^\gamma b'_n\}\,\rho^{2n-2}\cos 2n\theta\right] \\ \widehat{\theta\theta} &= \frac{M}{b^2}\sum_{n=1}^{\infty}\{2n(2n-1)\,{}^\gamma a'_n + (2n+1)(2n+2)\,{}^\gamma b'_n \rho^2\}\,\rho^{2n-2}\sin 2n\theta \end{aligned}\right\}, \quad (45)$$

giving at the rim of the hole

$$\left.\begin{aligned} \widehat{rr} &= \frac{M}{b^2}\sum_{n=1}^{\infty} {}^\gamma p_n \sin 2n\theta \\ \widehat{r\theta} &= \frac{M}{b^2}\sum_{n=0}^{\infty} {}^\gamma r_n \cos 2n\theta \end{aligned}\right\}, \quad \ldots \ldots \ldots \quad (46)$$

where

$$\left.\begin{aligned} {}^\gamma p_n &= -\{2n(2n-1)\,{}^\gamma a'_n + (2n-2)(2n+1)\,{}^\gamma b'_n \lambda^2\}\,\lambda^{2n-2} \\ {}^\gamma r_n &= -\{2n(2n-1)\,{}^\gamma a'_n + 2n(2n+1)\,{}^\gamma b'_n \lambda^2\}\,\lambda^{2n-2}, \quad n>0 \\ {}^\gamma r_0 &= -\frac{1}{2\pi\lambda^2} \end{aligned}\right\}. \quad \ldots \quad (47)$$

The solutions for isolated forces at the origin are not fully determinate unless the infinity conditions are specified, since otherwise any solution of type χ' may be added without altering either the boundary conditions, or the nature of the singularity at the origin. It is therefore essential to know the nature of the stresses at infinity given by χ_α, χ_β, χ_γ. For this purpose the series are useless, as their radius of convergence is finite (see p. 175). A direct derivation of the infinity conditions from the integral forms of χ_α and χ_β has been given elsewhere ("B," pp. 100, 104). It will be necessary, however, to carry the discussion of χ_β a little further, and also to deal with χ_γ.

It was shown that χ_α, in the form given, corresponds to a force P at the origin balanced by a uniform thrust $P/4b$ at $+\infty$ and a uniform tension $P/4b$ at $-\infty$. Calculation showed that the local effects of the force disappeared so rapidly as x increased that, for practical purposes, "infinity" might be taken to mean $\xi > 2$, *i.e.*, $x > 2b$. Either $+\infty$ or $-\infty$ may be freed from stress by the addition of a tension term from χ'. It was also shown that the principal part of the stresses at infinity due to χ_β consisted of a shear $-\frac{1}{2}P$ and a bending-moment $-\frac{1}{2}Px$, the asymptotic values of the stresses being

$$\widehat{xx} \sim -\frac{3Pxy}{4b^3}, \quad \widehat{yy} \sim 0, \quad \widehat{xy} \sim -\frac{3P}{8b^3}(y^2 - b^2).$$

This result does not, however, exclude the possibility of there being an additional finite bending-moment. To decide this point we write Φ_β in the form

$$\begin{aligned}\Phi_\beta = &\int_0^\infty \left\{\frac{u\eta c\,C - (c + us)\,S}{u^2\,\Sigma'} - \frac{\eta(\eta^2 - 3)}{4u^2}\right\}\{2(1-\sigma) + u\}\,e^{-u}\cos u\,\xi\,du \\ &+ \int_0^\infty \left\{\frac{\eta s\,C - c\,S}{u\,\Sigma'} - \frac{\eta(\eta^2-1)}{4u}\right\}\{(1-2\sigma) + u\}\,e^{-u}\cos u\,\xi\,du \\ &- \int_0^\infty \left[\frac{\eta(\eta^2-3)}{4u^2}\{2(1-\sigma) + u\} + \frac{\eta(\eta^2-1)}{4u}\{(1-2\sigma)+u\}\right] \\ &\qquad\qquad .\,(1 - \cos u\,\xi)\,e^{-u}\,du \\ = &\int_0^\infty f(u)\cos u\,\xi\,du - \frac{\eta}{4}\int_0^\infty (a + bu + cu^2)\,\frac{1-\cos u\,\xi}{u^2}\,e^{-u}du,\end{aligned}$$

where

$$\left.\begin{aligned} a &= 2(1-\sigma)(\eta^2 - 3) \\ b &= 2(1-\sigma)\,\eta^2 - 2(2-\sigma) \\ c &= \eta^2 - 1\end{aligned}\right\}.$$

The value of $f(u)$ need not be written down, for it is easily seen that the first term of Φ_β contributed to each of the stresses an integral of one of the types $\int_0^\infty \phi(u)\cos u\,\xi\,du$, $\int_0^\infty \phi(u)\sin u\,\xi\,du$, where in each case $\phi(0)$ is finite, and the values of such integrals tend to 0 when $\xi \to \infty$. The stresses at infinity are therefore contributed entirely by the second integral in Φ_β. To evaluate this, we start with

$$\int_0^\infty e^{-u} \cos u\,\xi\, du = \frac{1}{1+\xi^2}.$$

Integrating twice with respect to ξ under the sign of integration,

$$\int_0^\infty e^{-u} \frac{\sin u\,\xi}{u}\, du = \int_0^\xi \frac{d\,\xi}{1+\xi^2} = \tan^{-1}\xi,$$

$$\int_0^\infty e^{-u} \frac{1-\cos u\,\xi}{u^2}\, du = \int_0^\xi \tan^{-1}\xi\, d\,\xi = \xi \tan^{-1}\xi - \tfrac{1}{2}\log(1+\xi^2).$$

Similarly, from

$$\int_0^\infty e^{-u} \sin u\,\xi\, du = \frac{\xi}{1+\xi^2},$$

we find

$$\int_0^\infty e^{-u} \frac{1-\cos u\,\xi}{u}\, du = \int_0^\xi \frac{\xi}{1+\xi^2}\, d\,\xi = \tfrac{1}{2}\log(1+\xi^2).$$

Hence

$$\int_0^\infty (a+bu+cu^2)\frac{1-\cos u\,\xi}{u^2}\, e^{-u}\, du$$
$$= 2(1-\sigma)(\eta^2-3)\,\xi\tan^{-1}\xi + (1-2\sigma)\log(1+\xi^2) + \frac{(\eta^2-1)\xi^2}{1+\xi^2}.$$

The second term is of lower order than the others and may be omitted. In the first, $\tan^{-1}\xi$ differs from $\pi/2$ by a quantity of order $1/\xi^2$, and may be replaced by its asymptotic value; in the last term $\xi^2/(1+\xi^2)$ may be replaced by 1. Hence

$$\chi_\beta \sim -\frac{Pb}{4\pi(1-\sigma)}\left[\tfrac{1}{2}\pi(1-\sigma)\eta(\eta^2-3)\xi + \tfrac{1}{2}\eta(\eta^2-1)\right].$$

The corresponding stresses are

$$\widehat{xx} \sim -\frac{3P}{4b}\xi\eta - \frac{3P}{4\pi b(1-\sigma)}\eta,$$

$$\widehat{yy} \sim 0, \quad \widehat{xy} \sim -\frac{3P}{8b}(\eta^2-1).$$

Integrating across the strip we get a bending-moment $-\tfrac{1}{2}Px - \dfrac{P}{2\pi(1-\sigma)}$ and a shear $-\tfrac{1}{2}P$. The constant bending-moment $-P/2\pi(1-\sigma)$ must be taken into account when the infinity conditions are adjusted by a proper choice of χ'.

The discussion of χ_γ is simpler, for Φ_γ has the form $\int_0^\infty f(u) \sin u\,\xi\, du$. The limiting value of this when $\xi \to \infty$ is known to be $\frac{\pi}{2} \lim_{u \to 0} \{u f(u)\}$.* Now

$$f(u) = \frac{1}{2\Sigma'}[\eta C\{2u - (5 - 4\sigma) + e^{-2u}\}$$

$$+ \frac{S}{u}\{2(1-\sigma) + 4(1-\sigma)u - 2u^2 + 2(1-\sigma)e^{-2u}\}]$$

$$= \frac{2\eta(1-\sigma)}{\Sigma'}(\tfrac{1}{3}\eta^2 + 1)u^2 + O(u^4).$$

But $\Sigma' = \frac{4}{3}u^3 + O(u^5)$;

therefore

$$f(u) = \tfrac{1}{2}(1-\sigma)\eta(\eta^2+3)\frac{1}{u} + O(u)$$

and

$$\Phi_\gamma \sim \frac{\pi}{4}(1-\sigma)\eta(\eta^2+3).$$

This gives $\chi_\gamma \sim \frac{1}{8} M\eta(\eta^2 + 3)$, with the stresses

$$\widehat{xx} \sim \frac{3M\eta}{4b}, \quad \widehat{xy} \sim \widehat{yy} \sim 0.$$

Integration across the strip gives a bending-moment $\frac{1}{2}$M. Since χ_γ is odd in x, there will be a bending-moment $-\frac{1}{2}$ M at $-\infty$.

To complete the discussion of χ'' we need first to evaluate the coefficients in the expansions and then to establish the convergence of the series used. For these purposes a discussion, and an evaluation of the integrals I_s, J_s, I'_s, J'_s is needed.

§ 4. The Integrals I_s, J_s, I'_s, J'_s.

The integrals I_s and J_s have been discussed in a previous paper ("A," pp. 61–67), and their values have been calculated. The asymptotic values of the integrals were found to be

$$I_s \sim \frac{s!}{2^s}, \; J_s \sim \frac{s!}{2^{2s+1}}, \qquad (48)$$

and it was shown that, within the limits of five-figure tabulation, I_s agreed with its asymptotic value when $s > 20$. J_s becomes negligible in comparison with I_s when $s > 16$. The numerical values will not be repeated here, but we desire to record minor corrections to two of them. The corrected values are

$$I_{11} = 1 \cdot 9436 \times 10^4, \quad I_{14} = 5 \cdot 3185 \times 10^6.$$

* *Cf.* Carslaw, "Fourier Series and Integrals," London, p. 219 (1930).

Turning now to I'_s and J'_s, we begin with three simple lemmas, as follows :—

(i) u^3/Σ' is, for positive values of u, a monotonically decreasing function, with a value $< \frac{3}{4}$.

(ii) If $u \geq 0\cdot 5$, $e^{-2u} < u$.

(iii) If $u \geq 0\cdot 5$, $ue^{-2u} < 0\cdot 2$.

To prove (i) we write

$$\frac{d}{du}\left[\frac{u^3}{\Sigma'}\right] = \left(\frac{u}{\Sigma'}\right)^2 (3 \sinh 2u - 2u \cosh 2u - 4u)$$

$$= \left(\frac{u}{\Sigma'}\right)^2 \left\{- 3u - \sum_{n=1}^{\infty} \left[\frac{2}{(2n)\,!} - \frac{3}{(2n+1)\,!}\right] u^{2n+1}\right\}$$

$$= -\left(\frac{u}{\Sigma'}\right)^2 \left\{3u + \sum_{n=1}^{\infty} \frac{4n-1}{(2n+1)\,!} u^{2n+1}\right\}$$

$$< 0 \text{ if } u \text{ is positive.}$$

But

$$\lim_{u \to 0} \frac{u^3}{\Sigma'} = \tfrac{3}{4};$$

therefore

$$\frac{u^3}{\Sigma'} \leq \tfrac{3}{4} \text{ for } u \geq 0.$$

Obviously (ii) will be true in general if it is true when $u = 0\cdot 5$, and this needs no proof, since $\frac{1}{e} < \frac{1}{2}$.

For (iii) it is sufficient to write

$$\frac{d}{du}(ue^{-2u}) = e^{-2u}(1 - 2u) < 0 \text{ if } u > \tfrac{1}{2}.$$

Hence, when $u > \frac{1}{2}$, ue^{-2u} steadily decreases. But when $u = \frac{1}{2}$, $ue^{-2u} = \frac{1}{2e} < \frac{1}{5}$; this proves (iii).

An upper limit to I'_s may now be found by dividing the range of integration into two parts, 0 to $0\cdot 5$ and $0\cdot 5$ to ∞. In the lower range, using lemma (i),

$$\int_0^{0\cdot 5} \frac{u^s}{\Sigma'}\, du < \tfrac{3}{4} \int_0^{0\cdot 5} u^{s-3} du = \frac{3}{2^s(s-2)}.$$

In the upper range write

$$\Sigma' = \tfrac{1}{2} e^{2u} \{1 - e^{-4u} - 4ue^{-2u}\}$$

$$> \tfrac{1}{2} e^{2u} \{1 - ue^{-2u} - 4ue^{-2u}\}, \text{ by lemma (ii)},$$

$$= \tfrac{1}{2} e^{2u} (1 - 5ue^{-2u}).$$

By lemma (iii), the right-hand side is positive ; hence

$$\frac{1}{\Sigma'} < 2e^{-2u}(1 - 5ue^{-2u})^{-1}$$

$$< 2e^{-2u}(1 + 5ue^{-2u}).$$

Therefore

$$\int_{0\cdot 5}^{\infty} \frac{u^s}{\Sigma'}\,du < 2\int_0^{\infty} e^{-2u}(1 + 5ue^{-2u})u^s du$$

$$= \frac{s\,!}{2^s} + \frac{5\,(s+1)\,!}{2^{2s+1}}.$$

Combining these results, we have

$$\mathrm{I}_s' < \frac{s\,!}{2^s}\left\{1 + \frac{5\,(s-1)}{2^{s+1}} + \frac{3}{(s-2)\,.\,s\,!}\right\}. \quad \quad (49)$$

To obtain a lower limit to I'_s we use the obvious inequalities, $0 < \Sigma' < \frac{1}{2}e^{2u}$. Then

$$\mathrm{I}'_s = \int_0^{\infty} \frac{u^s}{\Sigma'}\,du > 2\int_0^{\infty} u^s e^{-2u} du = 2\,\frac{s\,!}{2^{s+1}},$$

i.e.,

$$\mathrm{I}'_s > \frac{s\,!}{2^s}. \quad \quad (50)$$

From (49) and (50), we have

$$\mathrm{I}'_s \sim \frac{s\,!}{2^s} \quad \quad (51)$$

so that it has the same asymptotic value as I_s. It can also be seen that, if $s > 22$, $\mathrm{I}'_s = \frac{s\,!}{2^s}(1 + \alpha)$, where $\alpha < 10^{-5}$, *i.e.*, the asymptotic value may be taken as accurate within the limits of a five-figure table.

In the same way, it may be proved that

$$\mathrm{J}'_s \sim \frac{s\,!}{2^{2s+1}}. \quad \quad (52)$$

Closer limits for I'_s may be obtained as follows. We have

$$\mathrm{I}'_s = \int_0^{\infty} \frac{u^s}{\sinh 2u}\left\{1 - \frac{2u}{\sinh 2u}\right\}^{-1} du = \sum_{p=0}^{\nu} 2^p \int_0^{\infty} \frac{u^{s+p}}{\sinh^{p+1} 2u}\,du + \mathrm{R}_\nu,$$

where

$$\mathrm{R}_\nu = 2^{\nu+1}\int_0^{\infty} \frac{u^{s+\nu+1}}{\Sigma' \sinh^{\nu+1} 2u}\,du. \quad \quad (53)$$

Also

$$\int_0^\infty \frac{u^{s+p}}{\sinh^{p+1} 2u}\,du = 2^{p+1}\int_0^\infty u^{s+p}\,e^{-2(p+1)u}\{1-e^{-4u}\}^{-(p+1)}\,du$$
$$= 2^{p+1}\sum_{q=0}^{\infty}\frac{[p+1]_q}{q\,!}\int_0^\infty u^{s+p}\,e^{-2(p+2q+1)u}\,du$$
$$= \frac{(s+p)\,!}{2^s}\sum_{q=0}^{\infty}\frac{[p+1]_q}{q\,!\,(p+2q+1)^{s+p+1}},$$

where $[p+1]_q = (p+1)(p+2)\ldots(p+q+1)$ (*cf.* "A," p. 65). The convergence of this series may be proved by comparison with the series $\sum_{q=0}^{\infty}\frac{1}{q^{s+p+1}}$. When s is large, the convergence is rapid.

We now have

$$\mathrm{I}'_s = \sum_{p=0}^{\nu} 2^{p-s}(s+p)\,!\sum_{q=0}^{\infty}\frac{[p+1]_q}{q\,!\,(p+2q+1)^{s+p+1}} + \mathrm{R}_\nu. \qquad (54)$$

Dividing the range of integration in R_ν into two parts, we have in the lower range

$$2^{\nu+1}\int_0^{0.5}\frac{u^{s+\nu+1}}{\Sigma'\sinh^{\nu+1}2u}\,du = \int_0^{0.5}\frac{u^s}{\Sigma'}\left(\frac{2u}{\sinh 2u}\right)^{\nu+1}u^{s-3}\,du$$
$$< \tfrac{3}{4}\int_0^{0.5}u^{s-3}\,du = \frac{3}{2^s\,(s-2)}.$$

In the upper range, $\frac{1}{\Sigma'} < 2e^{-2u}\,(1+5ue^{-2u})$ and $\sinh 2u = \tfrac{1}{2}\,e^{2u}\,(1-e^{-4u})$

$$> \tfrac{1}{2}\,e^{2u}\,(1-ue^{-2u}),\text{ from lemma (ii),}$$

so that

$$\frac{1}{\sinh^{\nu+1}2u} < 2^{\nu+1}\,e^{-2(\nu+1)u}\,\{1+(\nu+1)\,ue^{-2u}\}.$$

Hence

$$2^{\nu+1}\int_{0.5}^\infty\frac{u^{s+\nu+1}\,du}{\Sigma'\sinh^{\nu+1}2u} < 2^{2\nu+3}\int_0^\infty u^{s+\nu+1}\,e^{-2(\nu+2)u}\{1+(\nu+6)ue^{-2u}\}\,du$$
$$= \frac{(s+\nu+1)\,!}{2^{s-\nu-2}}\left[\frac{1}{(\nu+2)^{s+\nu+1}} + \frac{(\nu+6)\,(s+\nu+2)}{2\,(\nu+3)^{s+\nu+2}}\right].$$

Combining these results,

$$\mathrm{R}_\nu < \frac{3}{2^s(s-2)} + \frac{(s+\nu+1)\,!}{2^{s-\nu-2}}\left[\frac{1}{(\nu+2)^{s+\nu+1}} + \frac{(\nu+6)\,(s+\nu+2)}{2\,(\nu+3)^{s+\nu+2}}\right]. \qquad (55)$$

This inequality is sufficient for most purposes, but, in order that the behaviour of R_ν when $\nu\to\infty$ may be studied, a slightly stronger inequality is needed. This is obtained by using in the range 0 to 0·5 the inequality

$$\left(\frac{2u}{\sinh 2u}\right)^{\nu+1} < \frac{1}{(1+\frac{4}{3}\,u^2)^{\nu+1}} < \frac{1}{1+\dfrac{4\,(\nu+1)}{3}\,u^2}.$$

Hence

$$2^{\nu+1}\int_0^{0\cdot5}\frac{u^{s+\nu+1}}{\Sigma'\sinh^{\nu+1}2u}\,du < \tfrac{3}{4}\int_0^{0\cdot5}\frac{u^{s-3}}{1+\dfrac{4(\nu+1)}{3}u^2}\,du$$

$$\leq \tfrac{3}{4}\int_0^{0\cdot5}\frac{du}{1+\dfrac{4(\nu+1)}{3}u^2} = \frac{3\sqrt{3}}{8\sqrt{\nu+1}}\tan^{-1}\sqrt{\frac{\nu+1}{3}}.$$

When $\nu\to\infty$, $\tan^{-1}\sqrt{\dfrac{\nu+1}{3}}\to\dfrac{\pi}{2}$, so that the integral tends to 0. The first term in (55) may thus be replaced by one tending to 0 when $\nu\to\infty$. In the second term of (55) we now use STIRLING's formula, writing

$$(s+\nu+1)\,!\sim\sqrt{2\pi(s+\nu+1)}\,(s+\nu+1)^{s+\nu+1}e^{-(s+\nu+1)}.$$

Then, retaining only the terms of highest order, we have

$$\frac{(s+\nu+1)\,!}{2^{s-\nu-2}}\frac{1}{(\nu+2)^{s+\nu+1}} = \mathrm{O}\left\{\nu^{\frac{1}{2}}e^{-(\nu+1)}\left(\frac{s+\nu+1}{\nu+2}\right)^{s+\nu+1}\right\}$$

But

$$\left(\frac{s+\nu+1}{\nu+2}\right)^{s+\nu+1} = \frac{1}{\left(1-\dfrac{s-1}{s+\nu+1}\right)^{s+\nu+1}}\sim\frac{1}{e^{-(s-1)}} = \mathrm{O}(1).$$

Hence the whole term $= \mathrm{O}(\nu^{\frac{1}{2}}e^{-\nu})$.

In the same way the last term in R_ν may be seen to be of order $\nu^{3/2}e^{-\nu}$. Hence R_ν tends to 0 when $\nu\to\infty$, and the series in (54) may be extended to infinity.

On the other hand, since $\Sigma' < \frac{1}{2}e^{2u}$ and $\sinh 2u < \frac{1}{2}e^{2u}$,

$$\mathrm{R}_\nu > 2^{2\nu+3}\int_0^\infty u^{s+\nu+1}e^{-2(\nu+2)u}\,du = 2^{\nu-s+2}\frac{(s+\nu+1)\,!}{(\nu+2)^{s+\nu+1}}, \quad \ldots\ldots \quad (56)$$

and, if s is small, ν must be large before R_ν becomes small. Even with $s=8$ it will be found that ν must be taken >10 if R_ν is not to affect the fifth significant figure in the value of I'_s. For larger values of s, however, (54) gives a rapid method of computing I'_s. For example, if $s>13$, I'_s is given, to five significant figures by

$$\mathrm{I}'_s = \frac{s\,!}{2^s}\left[1+\frac{s+1}{2^{s+1}}+\frac{4(s+1)(s+2)}{3^{s+3}}\right] \quad \ldots\ldots\ldots \quad (57)$$

and in this the last term is negligible if $s>15$, and the second term if $s>21$. The values of I'_s for values of $s\geq 14$ in Table I were obtained in this way. Those for values of s from 3 to 18 were obtained by direct computation. The integrands were tabulated for values of u from 0 to 9 at intervals of 0·25, VAN ORSTRAND's tables of the exponential function being used.* The integrals over this range were then computed by WEDDLE's Rule.† The integrals from $u=9$ to ∞ were computed by replacing Σ' by $\frac{1}{2}e^{2u}$; they

* 'Mem. Nat. Acad. Sc.,' vol. 14 (1925).

† WHITTAKER and ROBINSON, "The Calculus of Observations," § 75.

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then become elementary. For $s = 14$ to 18 both methods were used, the results being in agreement.

Formulæ of the types (54) and (57) may be found for J'_s, but they only become of value for computing the integrals when s has fairly large values, and J'_s is then negligible in comparison with I'_s. All the values of J'_s given in Table I were found by direct computation. The ratios of I'_s and J'_s to $s!/2^s$ are also given.

From Table I the coefficients in χ_α, χ_β, χ_γ have been calculated for various values of λ and for $\sigma = \frac{1}{5}$. These appear in Table II. The value $\frac{1}{5}$ of σ corresponds to $\frac{1}{4}$ for Poisson's Ratio, which is roughly correct for most metals. If the coefficients should be required for other values of σ, their calculation will present no difficulty.

Table I.—Values of the Integrals I'_s, J'_s.

s.	I'_s.	$2^s I'_s/s!$	J'_s.	$2^s J'_s/s!$
1	$(1 \cdot 6042 \times 10^{-1})$*	$(0 \cdot 32084)$*	$(2 \cdot 1804 \times 10^{-1})$*	$(0 \cdot 43608)$*
2	$(8 \cdot 3338 \times 10^{-1})$*	$(1 \cdot 66676)$*	$(-5 \cdot 5286 \times 10^{-1})$*	$(-1 \cdot 10572)$*
3	$1 \cdot 5290 \times 10^{0}$	$2 \cdot 03867$	$3 \cdot 4554 \times 10^{-1}$	$0 \cdot 46072$
4	$2 \cdot 0299 \times 10^{0}$	$1 \cdot 35327$	$1 \cdot 4897 \times 10^{-1}$	$0 \cdot 09931$
5	$4 \cdot 3382 \times 10^{0}$	$1 \cdot 15685$	$1 \cdot 2157 \times 10^{-1}$	$0 \cdot 03242$
6	$1 \cdot 2113 \times 10^{1}$	$1 \cdot 07671$	$1 \cdot 419 \times 10^{-1}$	$0 \cdot 01261$
7	$4 \cdot 0920 \times 10^{1}$	$1 \cdot 03924$	$2 \cdot 123 \times 10^{-1}$	$0 \cdot 00539$
8	$1 \cdot 6073 \times 10^{2}$	$1 \cdot 02051$	$3 \cdot 832 \times 10^{-1}$	$0 \cdot 00243$
9	$7 \cdot 1645 \times 10^{2}$	$1 \cdot 01086$	$8 \cdot 05 \times 10^{-1}$	$0 \cdot 00114$
10	$3 \cdot 5642 \times 10^{3}$	$1 \cdot 00577$	$1 \cdot 92 \times 10^{0}$	$0 \cdot 00054$
11	$1 \cdot 9551 \times 10^{4}$	$1 \cdot 00310$	$5 \cdot 12 \times 10^{0}$	$0 \cdot 00026$
12	$1 \cdot 1714 \times 10^{5}$	$1 \cdot 00168$	$1 \cdot 5 \times 10^{1}$	$0 \cdot 00013$
13	$7 \cdot 6080 \times 10^{5}$	$1 \cdot 00088$	$4 \cdot 7 \times 10^{1}$	$0 \cdot 00006$
14	$5 \cdot 3234 \times 10^{6}$	$1 \cdot 00046$	$1 \cdot 7 \times 10^{2}$	$0 \cdot 00003$
15	$3 \cdot 9917 \times 10^{7}$	$1 \cdot 00025$	$6 \cdot 2 \times 10^{2}$	$0 \cdot 00002$
16	$3 \cdot 1930 \times 10^{8}$	$1 \cdot 00013$	$2 \cdot 5 \times 10^{3}$	$0 \cdot 00001$
17	$2 \cdot 7139 \times 10^{9}$	$1 \cdot 00007$	$1 \cdot 0 \times 10^{4}$	—
18	$2 \cdot 4424 \times 10^{10}$	$1 \cdot 00004$	$4 \cdot 5 \times 10^{4}$	—
19	$2 \cdot 3202 \times 10^{11}$	$1 \cdot 00001$	2×10^{5}	—
20	$2 \cdot 3202 \times 10^{12}$	$1 \cdot 00001$	1×10^{6}	—
21	$2 \cdot 4362 \times 10^{13}$	$1 \cdot 00000$	—	—

* See special definitions, equations (28).

Table II.—Coefficients in χ_α, χ_β and χ_γ $(\sigma = 0 \cdot 2)$.

n.	${}^{\alpha}a'_n$.	${}^{\alpha}b'_n$.	${}^{\beta}a'_n$.	${}^{\beta}b'_n$.	${}^{\gamma}a'_n$.	${}^{\gamma}b'_n$.
0	—	$-4 \cdot 667 \times 10^{-3}$	—	$1 \cdot 835 \times 10^{-2}$	—	—
1	$-1 \cdot 488 \times 10^{-2}$	$2 \cdot 594 \times 10^{-3}$	$-6 \cdot 174 \times 10^{-2}$	$1 \cdot 610 \times 10^{-2}$	$7 \cdot 089 \times 10^{-2}$	$-1 \cdot 672 \times 10^{-2}$
2	$-1 \cdot 547 \times 10^{-3}$	$9 \cdot 526 \times 10^{-4}$	$+1 \cdot 038 \times 10^{-3}$	$2 \cdot 341 \times 10^{-3}$	$1 \cdot 836 \times 10^{-2}$	$2 \cdot 540 \times 10^{-3}$
3	$-2 \cdot 645 \times 10^{-4}$	$2 \cdot 784 \times 10^{-4}$	$-1 \cdot 799 \times 10^{-4}$	$5 \cdot 079 \times 10^{-4}$	$3 \cdot 567 \times 10^{-3}$	$1 \cdot 478 \times 10^{-3}$
4	$-4 \cdot 937 \times 10^{-5}$	$7 \cdot 546 \times 10^{-5}$	$-4 \cdot 397 \times 10^{-5}$	$1 \cdot 193 \times 10^{-5}$	$8 \cdot 889 \times 10^{-4}$	$5 \cdot 647 \times 10^{-4}$
5	$-9 \cdot 816 \times 10^{-6}$	$1 \cdot 981 \times 10^{-5}$	$-9 \cdot 406 \times 10^{-6}$	$2 \cdot 879 \times 10^{-5}$	$2 \cdot 298 \times 10^{-4}$	$1 \cdot 896 \times 10^{-4}$
6	$-2 \cdot 033 \times 10^{-6}$	$5 \cdot 115 \times 10^{-6}$	$-2 \cdot 002 \times 10^{-6}$	$7 \cdot 029 \times 10^{-6}$	$5 \cdot 945 \times 10^{-5}$	$5 \cdot 952 \times 10^{-5}$
7	$-4 \cdot 344 \times 10^{-7}$	$1 \cdot 309 \times 10^{-6}$	$-4 \cdot 321 \times 10^{-7}$	$1 \cdot 727 \times 10^{-6}$	$1 \cdot 529 \times 10^{-5}$	$1 \cdot 791 \times 10^{-5}$

§ 5. Convergence of the Series for χ_α, χ_β, χ_γ.

From (13) and (14) and from the asymptotic values of the integrals I_n and J_n, it is clear that the asymptotic values of the coefficients ${}^\alpha a_n$, ${}^\alpha b_n$ are

$$ {}^\alpha a_n \sim -\frac{1}{(2n+1)!}\frac{(2n+1)!}{2^{2n+1}} = -\frac{1}{2^{2n+1}}, $$

$$ {}^\alpha b_n \sim \frac{1}{2(2n+2)!}\frac{(2n+2)!}{2^{2n+2}} = \frac{1}{2^{2n+3}}, $$

from which it follows at once that the series for χ_α converges within the circle $\rho = 2$. The same may be proved for χ_β and χ_γ, and for the series for the stresses derived from them. It is also obvious that the convergence is uniform within any circle $\rho = \rho_1 < 2$, so that the differentiations term by term were justified.

§ 6. The Construction of χ'''.

Since $\chi''' = \chi - \chi' - \chi''$ and the tractions due to χ are given in advance, it is now possible to write down those due to χ'''. The next step is to determine χ_0, the stress-function which would give the same tractions as χ''' if the plate were of infinite extent. The solution is then completed by writing

$$ \chi''' = \chi_0 + \chi_1 + \chi_2 + \chi_3 + \dots, \qquad (58) $$

where χ_1 cancels* the tractions produced by χ_0 on the straight boundaries, χ_2 cancels those produced by χ_1 on the circular boundary, and so on. The analysis will be divided into four parts, dealing with (i) stress-functions even in both co-ordinates ; (ii) functions odd in x and even in y ; (iii) functions even in x and odd in y ; (iv) functions odd in both x and y.

(i) χ''' *even in both x and y.*

The analysis for this case was given in the previous paper ("A," pp. 52–61). The results will be quoted, but with a slight change of notation, which becomes advisable in order that the formulæ in the other cases may be simplified.

χ''' is to satisfy the conditions

(i) $\widehat{rr}$, $\widehat{r\theta}$, $\widehat{\theta\theta}$ all tend to 0 when $\xi \to 0$;

(ii) $\widehat{yy} = \widehat{xy} = 0$ when $\eta = \pm 1$;

* If χ is to give non-zero tractions over the straight edges, χ_1 must provide these, in addition to cancelling the tractions produced by χ_0 over those boundaries.

(iii) $\widehat{rr} = f_1(\theta)$, $\widehat{r\theta} = f_2(\theta)$ when $\rho = \lambda$, $f_1(\theta)$ and $f_2(\theta)$ being known functions. Since χ''' is even in x and y, $f_1(\theta)$ and $f_2(\theta)$ may be expanded in the form

$$\left.\begin{aligned} f_1(\theta) &= \sum_{n=0}^{\infty} S_{2n} \cos 2n\theta \\ f_2(\theta) &= \sum_{n=1}^{\infty} T_{2n} \sin 2n\theta \end{aligned}\right\} \quad \ldots \ldots \quad (59)$$

Now write

$$\chi_0 = -D_0^{(0)} \log \rho + \sum_{n=1}^{\infty} \left(\frac{D_{2n}^{(0)}}{\rho^{2n}} + \frac{E_{2n}^{(0)}}{\rho^{2n-2}}\right) \cos 2n\theta. \quad \ldots \ldots \quad (60)$$

The corresponding stresses are given by

$$\widehat{rr} = \frac{1}{b^2}\left[-\frac{D_0^{(0)}}{\rho^2} - 2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}^{(0)}}{\rho^{2n+2}} + \frac{(n+1)(2n-1)E_{2n}^{(0)}}{\rho^{2n}}\right\}\cos 2n\theta\right],$$

$$\widehat{r\theta} = -\frac{2}{b^2}\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}^{(0)}}{\rho^{2n+2}} + \frac{n(2n-1)E_{2n}^{(0)}}{\rho^{2n}}\right\}\sin 2n\theta. \quad \ldots \ldots \quad (61)$$

These will be identical with the expansions of $f_1(\theta)$, $f_2(\theta)$ in (59) when $\rho = \lambda$, provided that

$$D_0^{(0)} = -b^2\lambda^2 S_0, \quad \ldots \ldots \quad (62)$$

$$2n(2n+1)D_{2n}^{(0)} + (2n-1)(2n+2)\lambda^2 E_{2n}^{(0)} = -b^2\lambda^{2n+2}S_{2n},$$

$$2n(2n+1)D_{2n}^{(0)} + 2n(2n-1)\lambda^2 E_{2n}^{(0)} = -b^2\lambda^{2n+2}T_{2n},$$

whence

$$\left.\begin{aligned} D_{2n}^{(0)} &= \frac{b^2\lambda^{2n+2}\{nS_{2n} - (n+1)T_{2n}\}}{2n(2n+1)} \\ E_{2n}^{(0)} &= \frac{b^2\lambda^{2n}(T_{2n} - S_{2n})}{2(2n-1)} \end{aligned}\right\} \quad \ldots \ldots \quad (63)^*$$

The solution is now completed by writing

$$\chi''' = \chi_0 + \chi_1 + \chi_2 + \chi_3 + \ldots, \quad \ldots \ldots \quad (58\ bis)$$

where

$$\left.\begin{aligned} \chi_{2r} &= -D_0^{(r)} \log \rho + \sum_{n=1}^{\infty}\left(\frac{D_{2n}^{(r)}}{\rho^{2n}} + \frac{E_{2n}^{(r)}}{\rho^{2n-2}}\right)\cos 2n\theta \\ \chi_{2r+1} &= \sum_{n=0}^{\infty}(L_{2n}^{(r)} + M_{2n}^{(r)}\rho^2)\rho^{2n}\cos 2n\theta \end{aligned}\right\} \quad \ldots \quad (64)$$

The coefficients in the successive stress-functions were shown to be related by the equations

$$\left.\begin{aligned} L_{2n}^{(r)} &= {}^{2n}\alpha_0 D_0^{(r)} + \sum_{p=1}^{\infty}\{{}^{2n}\alpha_{2p}D_{2p}^{(r)} + {}^{2n}\beta_{2p}E_{2p}^{(r)}\} \\ M_{2n}^{(r)} &= {}^{2n}\gamma_0 D_0^{(r)} + \sum_{p=1}^{\infty}\{{}^{2n}\gamma_{2p}D_{2p}^{(r)} + {}^{2n}\delta_{2p}E_{2p}^{(r)}\} \end{aligned}\right\} \quad \ldots \ldots \quad (65)\dagger$$

* Equations (62) and (63) were not given in "A."

† In "A" the coefficient here called ${}^{2n}\alpha_{2p}$ was written ${}^{n}\alpha_{p}$, and similarly for the other coefficients.

and

$$\left.\begin{aligned} D_0^{(r+1)} &= 2M_0^{(r)}\lambda^2 \\ D_{2n}^{(r+1)} &= \lambda^{4n}\{(2n-1)L_{2n}^{(r)} + 2n\lambda^2 M_{2n}^{(r)}\} \\ E_{2n}^{(r+1)} &= -\lambda^{4n-2}\{2nL_{2n}^{(r)} + (2n+1)\lambda^2 M_{2n}^{(r)}\} \end{aligned}\right\} \quad \ldots \quad (66)$$

In the notation now used, the coefficients in (65) have values included in the formulæ

$$\left.\begin{aligned} {}^n\alpha_0 &= -\frac{1}{n!}\{2I_n - (n-1)I_{n-1} - J_{n-1}\}, \\ {}^n\alpha_p &= -\frac{1}{n!\,(p-1)!}\{2I_{n+p} - (n-1)I_{n+p-1} - J_{n+p-1}\}, \\ {}^n\beta_p &= -\frac{1}{n!\,(p-2)!}\{4I_{n+p-1} - 2(n+p-2)I_{n+p-2} \\ &\qquad + (np-n-p+2)I_{n+p-3} + (n+p-2)J_{n+p-3}\}, \\ {}^n\gamma_0 &= \frac{I_{n+1}}{(n+1)!}, \qquad {}^n\gamma_p = \frac{I_{n+p+1}}{(n+1)!\,(p-1)!}, \\ {}^n\delta_p &= \frac{1}{(n+1)!\,(p-2)!}\{2I_{n+p} - (p-1)I_{n+p-1} - J_{n+p-1}\} \end{aligned}\right\} \quad . \quad (67)$$

The definitions apply not only to coefficients with even suffixes, but also to those with odd suffixes; these will be required in the next section. But ${}^n\beta_1$, ${}^n\delta_1$ are to be excluded; they will be defined later.

When $(n+p)$ is large the J integrals in (67) become negligible, while the I integrals may be replaced by their asymptotic values. After a little reduction, we then have the following asymptotic formulæ for the coefficients

$$\left.\begin{aligned} {}^n\alpha_0 &\sim -\frac{1}{n\,.\,2^{n-1}} \\ {}^n\alpha_p &\sim -\frac{(p+1)(n+p-1)!}{2^{n+p-1}n!\,(p-1)!} \\ {}^n\beta_p &\sim -\frac{p(n+p-3)!}{2^{n+p-3}(n-1)!\,(p-2)!} \\ {}^n\gamma_0 &\sim \frac{1}{2^{n+1}} \\ {}^n\gamma_p &\sim \frac{(n+p+1)!}{2^{n+p+1}(n+1)!\,(p-1)!} \\ {}^n\delta_p &\sim \frac{(n+p-1)!}{2^{n+p-1}n!\,(p-2)!} \end{aligned}\right\} \quad \ldots \quad (68)$$

The values of the coefficients for even suffixes up to 14 are given in Tables III to VI. Many of these were given previously.* A few have been slightly modified on account of the small corrections to the values of I_{11} and I_{14}. The changes are, however, all small, and affect only coefficients which, in the calculations previously made, multiplied small quantities. The stress-functions calculated in the previous paper are not affected to any appreciable extent by these slight corrections.

TABLE III.—Values of $- {}^n\alpha_p$ (even suffixes).

	$n = 2.$	$n = 4.$	$n = 6.$	$n = 8.$	$n = 10.$	$n = 12.$	$n = 14.$
$p = 0$	0·13674	0·031489	0·005656	0·001038	0·000201	0·000042	0·000009
$p = 2$	0·99865	0·45889	0·16450	0·052939	0·016146	0·004765	0·001374
$p = 4$	1·4962	1·3544	0·81924	0·40288	0·17460	0·06944	0·02594
$p = 6$	1·1319	1·7166	1·5779	1·0996	0·64151	0·33048	0·15524
$p = 8$	0·6298	1·4485	1·8848	1·7673	1·3354	0·86497	0·49902
$p = 10$	0·29497	0·95974	1·6801	2·0401	1·9382	1·5417	1·0716
$p = 12$	0·12373	0·54146	1·2274	1·8741	2·1865	2·0954	1·7269
$p = 14$	0·04806	0·27236	0·77625	1·4555	2·0459	2·3248	2·2417

TABLE IV.—Values of $- {}^n\beta_p$ (even suffixes).

	$n = 2.$	$n = 4.$	$n = 6.$	$n = 8.$	$n = 10.$	$n = 12.$	$n = 14.$
$p = 2$	0·96807	0·24555	0·063167	0·015777	0·003929	0·000979	0·000244
$p = 4$	1·4733	1·2367	0·65456	0·28134	0·10748	0·038093	0·01282
$p = 6$	0·94750	1·6364	1·4755	0·96675	0·52368	0·24995	0·10896
$p = 8$	0·44176	1·3129	1·8046	1·6757	1·2220	0·75544	0·41400
$p = 10$	0·17683	0·80607	1·5710	1·9638	1·8548	1·4416	0·97032
$p = 12$	0·064627	0·41903	1·0998	1·7807	2·1143	2·0183	1·6366
$p = 14$	0·02223	0·19442	0·66102	1·3455	1·9622	2·2565	2·1697

TABLE V.—Values of ${}^n\gamma_p$ (even suffixes).

	$n = 0.$	$n = 2.$	$n = 4.$	$n = 6.$	$n = 8.$	$n = 10.$	$n = 12.$	$n = 14.$
$p = 0$	0·38429	0·10346	0·028922	0·007599	0·001935	0·000487	0·000122	0·000031
$p = 2$	0·62078	0·57843	0·31918	0·13935	0·053566	0·019027	0·006407	0·002075
$p = 4$	0·57843	1·0639	0·97546	0·64279	0·34883	0·16658	0·072627	0·029572
$p = 6$	0·31918	0·97546	1·3499	1·2558	0·91621	0·56649	0·31050	0·15525
$p = 8$	0·13935	0·64279	1·2558	1·5706	1·4837	1·1533	0·77625	0·46760
$p = 10$	0·053566	0·34883	0·91621	1·4837	1·7620	1·6819	1·3638	0·97417
$p = 12$	0·019027	0·16658	0·56649	1·1533	1·6819	1·9342	1·8598	1·5542
$p = 14$	0·006407	0·07263	0·31050	0·77625	1·3638	1·8598	2·0922	2·0225

* "A," Tables II to V, pp. 68–69, but note the modified notation.

TABLE VI.—Values of ${}^n\delta_p$ (even suffixes).

	$n = 0.$	$n = 2.$	$n = 4.$	$n = 6.$	$n = 8.$	$n = 10.$	$n = 12.$	$n = 14.$
$p = 2$	0·27349	0·33288	0·14962	0·053899	0·017495	0·005363	0·001586	0·000458
$p = 4$	0·3779	0·9178	0·81267	0·49046	0·24141	0·10470	0·041652	0·015564
$p = 6$	0·1697	0·82250	1·2289	1·1271	0·78532	0·45818	0·23605	0·11089
$p = 8$	0·05811	0·49410	1·1281	1·4662	1·3746	1·0386	0·67276	0·38813
$p = 10$	0·01810	0·24220	0·78569	1·3747	1·6692	1·5858	1·2614	0·87675
$p = 12$	0·005434	0·10483	0·45826	1·0386	1·5858	1·8501	1·7731	1·4612
$p = 14$	0·001594	0·041670	0·23603	0·67274	1·2614	1·7731	2·0148	1·9428

At this point we modify the procedure formerly adopted by introducing formulæ leading directly from χ_{2n} to χ_{2n+2}. The new formulæ follow immediately from (65) and (66), and are

$$\left.\begin{aligned} D_{2n}^{(r+1)} &= {}^{2n}h_0 D_0^{(r)} + \sum_{p=1}^{\infty} \{{}^{2n}h_{2p} D_{2p}^{(r)} + {}^{2n}i_{2p} E_{2p}^{(r)}\} \\ E_{2n}^{(r+1)} &= {}^{2n}j_0 D_0^{(r)} + \sum_{p=1}^{\infty} \{{}^{2n}j_{2p} D_{2p}^{(r)} + {}^{2n}k_{2p} E_{2p}^{(r)}\} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (69)$$

where

$$\left.\begin{aligned} {}^n h_p &= \{(n-1)\,{}^n\alpha_p + n\lambda^2\,{}^n\gamma_p\}\,\lambda^{2n} \\ {}^n i_p &= \{(n-1)\,{}^n\beta_p + n\lambda^2\,{}^n\delta_p\}\,\lambda^{2n} \\ {}^n j_p &= -\{n\,{}^n\alpha_p + (n+1)\,\lambda^2\,{}^n\gamma_p\}\,\lambda^{2n-2} \\ {}^n k_p &= -\{n\,{}^n\beta_p + (n+1)\,\lambda^2\,{}^n\delta_p\}\,\lambda^{2n-2} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

except for $n = 0$, when

$${}^0h_0 = 2\lambda^2\,{}^0\gamma_0, \quad {}^0h_p = 2\lambda^2\,{}^0\gamma_p, \quad {}^0i_p = 2\lambda^2\,{}^0\delta_p, \quad {}^0j_p = {}^0k_p = 0. \quad . \quad . \quad . \quad (71)$$

The definitions in (70) apply also to the coefficients with odd suffixes, which will occur in the next section.

The values of the new coefficients for $\lambda = 0{\cdot}2,\ 0{\cdot}3,\ 0{\cdot}4,\ 0{\cdot}5$, have been calculated, but the tables are omitted on account of their length. With their aid the successive even stress-functions may be found. The odd functions need not be separately calculated, since only their sum is required and this may be found directly from the sum of those with even suffixes. For if

$$\left.\begin{aligned} d_{2n} &= \sum_r D_{2n}^{(r)}, \qquad e_{2n} = \sum_r E_{2n}^{(r)} \\ l_{2n} &= \sum_r L_{2n}^{(r)}, \qquad m_{2n} = \sum_r M_{2n}^{(r)} \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)^*$$

* The summations are over some finite number of values of r. The upper limit of summation is not taken as ∞ since the convergence of the series has not been proved.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Then it follows from (65) that

$$\left.\begin{aligned} l_{2n} &= {}^{2n}\alpha_0 d_0 + \sum_{p=1}^{\infty} \{{}^{2n}\alpha_{2p}\, d_{2p} + {}^{2n}\beta_{2p}\, e_{2p}\} \\ m_{2n} &= {}^{2n}\gamma_0 d_0 + \sum_{p=1}^{\infty} \{{}^{2n}\gamma_{2p}\, d_{2p} + {}^{2n}\delta_{2p}\, e_{2p}\} \end{aligned}\right\}, \quad \ldots \ldots \quad (73)$$

so that l_{2n}, m_{2n} are derivable from d_{2n} and e_{2n} with the aid of coefficients already tabulated.

In the calculations of which the results will be given in later sections, both methods of calculation have been used. The calculation is then self-checking. Each stress-function of even suffix is obtained twice, once directly from the previous even-suffixed function using (69), and once through the intermediary of an odd function, using (65) and (66). The total coefficients d_{2n}, e_{2n} are formed and the coefficients l_{2n}, m_{2n} are derived from (73); these are compared with the sums of the coefficients L_{2n}, M_{2n}. This double method of calculation is laborious, but it has the advantage that errors can hardly fail to be detected.

The process is carried on until it can be seen that the last even stress-function included is giving only very small residual tractions on the straight boundaries. Although it is highly probable that in most cases such a function will soon be reached, we cannot, in the absence of a definite proof, assume that the process is convergent. Such a proof we have not yet succeeded in constructing, although a good deal of progress towards it has been made. All the calculations that have been carried out in special cases have, however, confirmed our opinion that the convergence is rapid if $\lambda \leq 0 \cdot 5$.

The final stress-function is

$$\chi''' = -d_0 \log \rho + m_0\rho^2 + \sum_{n=1}^{\infty} \left\{\frac{d_{2n}}{\rho^{2n}} + \frac{e_{2n}}{\rho^{2n-2}} + (l_{2n} + m_{2n}\rho^2)\rho^{2n}\right\} \cos 2n\theta \quad . \; . \quad (74)$$

and the corresponding stress-components are

$$\left.\begin{aligned} \widehat{rr} &= 2m_0 - \frac{d_0}{\rho^2} - 2\sum_{n=1}^{\infty} \left\{\frac{n(2n+1)d_{2n}}{\rho^{2n+2}} + \frac{(n+1)(2n-1)\,e_{2n}}{\rho^{2n}} \right. \\ &\qquad \left. + n\,(2n-1)\,l_{2n}\rho^{2n-2} + (n-1)\,(2n+1)\,m_{2n}\rho^{2n}\right\} \cos 2n\theta \\ \widehat{\theta\theta} &= 2m_0 + \frac{d_0}{\rho^2} + 2\sum_{n=1}^{\infty} \left\{\frac{n(2n+1)d_{2n}}{\rho^{2n+2}} + \frac{(n-1)(2n-1)\,e_{2n}}{\rho^{2n}} \right. \\ &\qquad \left. + n\,(2n-1)\,l_{2n}\rho^{2n-2} + (n+1)\,(2n+1)\,m_{2n}\rho^{2n}\right\} \cos 2n\theta \\ \widehat{r\theta} &= 2\sum_{n=1}^{\infty} \left\{n\,(2n-1)\left(l_{2n}\rho^{2n-2} - \frac{e_{2n}}{\rho^{2n}}\right) + n\,(2n+1)\left(m_{2n}\,\rho^{2n} - \frac{d_{2n}}{\rho^{2n+2}}\right)\right\} \sin 2n\theta \end{aligned}\right\} . \, (75)$$

From these and appropriate transformation formulæ, the Cartesian stress-components are readily found. The transformation formulæ are given below in equations (81).

(ii) χ''' *odd in x and even in y.*

Let the tractions at $\rho = \lambda$ be $\widehat{rr} = f_3(\theta)$, $\widehat{r\theta} = f_4(\theta)$, these being known functions of θ having expansions of the forms

$$\left.\begin{aligned} f_3(\theta) &= \sum_{n=0}^{\infty} U_{2n+1} \sin(2n+1)\theta \\ f_4(\theta) &= \sum_{n=0}^{\infty} V_{2n+1} \cos(2n+1)\theta \end{aligned}\right\} \qquad (76)$$

Then, as before, we find

$$\chi_0 = \sum_{n=0}^{\infty} \left(\frac{F^{(0)}_{2n+1}}{\rho^{2n+1}} + \frac{G^{(0)}_{2n+1}}{\rho^{2n-1}} \right) \sin(2n+1)\theta, \qquad (77)$$

where

$$\left.\begin{aligned} F^{(0)}_{2n+1} &= \frac{b^2 \lambda^{2n+3} \{(2n+1) U_{2n+1} + (2n+3) V_{2n+3}\}}{4(n+1)(2n+1)} \\ G^{(0)}_{2n+1} &= -\frac{b^2 \lambda^{2n+1} (U_{2n+1} + V_{2n+1})}{4n} \end{aligned}\right\} \qquad (78)$$

Let

$$\chi''' = \chi_0 + \chi_1 + \chi_2 + \chi_3 + \ldots, \qquad (58\ bis)$$

where

$$\chi_{2r} = \sum_{n=0}^{\infty} \left(\frac{F^{(r)}_{2n+1}}{\rho^{2n+1}} + \frac{G^{(r)}_{2n+1}}{\rho^{2n-1}} \right) \sin(2n+1)\theta. \qquad (79)$$

For simplicity of writing, the suffix (r) will be omitted for the present, there being no immediate possibility of ambiguity. The stress-components may then be written

$$\left.\begin{aligned} \widehat{rr} &= -\frac{2}{b^2} \sum_{n=0}^{\infty} \left\{ \frac{(n+1)(2n+1) F_{2n+1}}{\rho^{2n+3}} + \frac{n(2n+3) G_{2n+1}}{\rho^{2n+1}} \right\} \sin(2n+1)\theta \\ \widehat{\theta\theta} &= \frac{2}{b^2} \sum_{n=0}^{\infty} \left\{ \frac{(n+1)(2n+1) F_{2n+1}}{\rho^{2n+3}} + \frac{n(2n-1) G_{2n+1}}{\rho^{2n+1}} \right\} \sin(2n+1)\theta \\ \widehat{r\theta} &= \frac{2}{b^2} \sum_{n=0}^{\infty} \left\{ \frac{(n+1)(2n+1) F_{2n+1}}{\rho^{2n+3}} + \frac{n(2n+1) G_{2n+1}}{\rho^{2n+1}} \right\} \cos(2n+1)\theta \end{aligned}\right\} \qquad (80)$$

When the Cartesian and polar co-ordinates are related, as in (1), the Cartesian stress-components are given by ("A" p. 53)

$$\left.\begin{aligned} \widehat{xx} &= \tfrac{1}{2} \{(\widehat{rr} + \widehat{\theta\theta}) - (\widehat{rr} - \widehat{\theta\theta}) \cos 2\theta\} + \widehat{r\theta} \sin 2\theta \\ \widehat{yy} &= \tfrac{1}{2} \{(\widehat{rr} + \widehat{\theta\theta}) + (\widehat{rr} - \widehat{\theta\theta}) \cos 2\theta\} - \widehat{r\theta} \sin 2\theta \\ \widehat{xy} &= \tfrac{1}{2} (\widehat{rr} - \widehat{\theta\theta}) \sin 2\theta + \widehat{r\theta} \cos 2\theta \end{aligned}\right\} \qquad (81)$$

From (80)

$$\tfrac{1}{2}(\widehat{rr}-\widehat{\theta\theta}) = -\frac{2}{b^2}\sum_{n=0}^{\infty}\left\{\frac{(n+1)(2n+1)F_{2n+1}}{\rho^{2n+3}}+\frac{n(2n+1)G_{2n+1}}{\rho^{2n+1}}\right\}\sin(2n+1)\theta$$

$$\tfrac{1}{2}(\widehat{rr}+\widehat{\theta\theta}) = -\frac{4}{b^2}\sum_{n=0}^{\infty}\frac{n\,G_{2n+1}}{\rho^{2n+1}}\sin(2n+1)\theta.$$

Therefore

$$\tfrac{1}{2}(\widehat{rr}-\widehat{\theta\theta})\cos 2\theta-\widehat{r\theta}\sin 2\theta = -\frac{2}{b^2}\sum_{n=0}^{\infty}\left\{\frac{(n+1)(2n+1)F_{2n+1}}{\rho^{2n+3}}+\frac{n(2n+1)G_{2n+1}}{\rho^{2n+1}}\right\}\sin(2n+3)\theta$$

and

$$\tfrac{1}{2}(\widehat{rr}-\widehat{\theta\theta})\sin 2\theta+\widehat{r\theta}\cos 2\theta = \frac{2}{b^2}\sum_{n=0}^{\infty}\left\{\frac{(n+1)(2n+1)F_{2n+1}}{\rho^{2n+3}}+\frac{n(2n+1)G_{2n+1}}{\rho^{2n+1}}\right\}\cos(2n+3)\theta.$$

Substitution into (81) now gives

$$\left.\begin{aligned}\widehat{xx} &= \frac{2}{b^2}\sum_{n=1}^{\infty}\left[\frac{n\{(2n-1)F_{2n-1}-2G_{2n+1}\}}{\rho^{2n+1}}+\frac{(n-1)(2n-1)G_{2n-1}}{\rho^{2n-1}}\right]\sin(2n+1)\theta\\ \widehat{yy} &= -\frac{2}{b^2}\sum_{n=1}^{\infty}\left[\frac{n\{(2n-1)F_{2n-1}+2G_{2n+1}\}}{\rho^{2n+1}}+\frac{(n-1)(2n-1)G_{2n-1}}{\rho^{2n-1}}\right]\sin(2n+1)\theta\\ \widehat{xy} &= \frac{2}{b^2}\sum_{n=1}^{\infty}\left[\frac{n(2n-1)F_{2n-1}}{\rho^{2n+1}}+\frac{(n-1)(2n-1)G_{2n-1}}{\rho^{2n-1}}\right]\cos(2n+1)\theta\end{aligned}\right\}. \quad \ldots \quad (82)$$

We have now to form χ_{2r+1} so that the tractions $\widehat{yy}$ and $\widehat{xy}$ in (82) are cancelled on the edges $\eta=\pm 1$. The tractions produced by χ_{2r+1} on $\eta=1$ are thus obtained from (82) by putting $\eta=1$, $\rho^2=1+\xi^2$, and then changing the sign. They are

$$\left.\begin{aligned}\phi(\xi)=\widehat{yy} &= \frac{2}{b^2}\sum_{n=1}^{\infty}\left[\frac{n\{(2n-1)F_{2n-1}+2G_{2n+1}\}}{(1+\xi^2)^{\frac{2n+1}{2}}}+\frac{(n-1)(2n-1)G_{2n-1}}{(1+\xi^2)^{\frac{2n-1}{2}}}\right]\sin(2n+1)\theta\\ \psi(\xi)=\widehat{xy} &= -\frac{2}{b^2}\sum_{n=1}^{\infty}\left[\frac{n(2n-1)F_{2n-1}}{(1+\xi^2)^{\frac{2n+1}{2}}}+\frac{(n-1)(2n-1)G_{2n-1}}{(1+\xi^2)^{\frac{2n-1}{2}}}\right]\cos(2n+1)\theta\end{aligned}\right\} \quad \ldots \quad (83)$$

where now

$$\tan\theta=\xi \quad \ldots \quad (84)$$

$\widehat{yy}$ is odd in x and even in y; $\widehat{xy}$ is even in x and odd in y.

χ_{2r+1} is now given by ("B," p. 93)

$$\chi_{2r+1}=\chi'_{2r+1}+\chi''_{2r-1} \quad \ldots \quad (85)$$

where

$$\left.\begin{aligned}\chi'_{2r+1} &= \frac{4b^2}{\pi}\int_0^\infty \frac{u\eta s\mathrm{S} - (s + uc)\,\mathrm{C}}{u^2\Sigma}\sin u\,\xi\,du\int_0^\infty \phi\,(w)\sin uw\,dw \\ \chi''_{2r+1} &= \frac{4b^2}{\pi}\int_0^\infty \frac{s\mathrm{C} - \eta c\mathrm{S}}{u\Sigma}\sin u\,\xi\,du\int_0^\infty \psi\,(w)\cos uw\,dw\end{aligned}\right\} \quad . \; . \; . \; . \; (86)$$

$\phi\,(w)$ and $\psi\,(w)$ being obtained from (83) and (84) by changing the variable.

Changing n to $n \pm \frac{1}{2}$ in the integral formulæ previously established,* we have

$$\left.\begin{aligned}\int_0^\infty \frac{\cos\,(2n+1)\,\theta\cos uw}{(1+w^2)^{\frac{2n+1}{2}}}\,dw &= \int_0^\infty \frac{\sin\,(2n+1)\,\theta\sin uw}{(1+w^2)^{\frac{2n+1}{2}}}\,dw = \frac{\pi\,e^{-u}\,u^{2n}}{2\,.\,(2n)!} \\ \int_0^\infty \frac{\cos\,(2n+1)\,\theta\cos uw}{(1+w^2)^{\frac{2n-1}{2}}}\,dw &= \int_0^\infty \frac{\sin\,(2n+1)\,\theta\sin uw}{(1+w^2)^{\frac{2n-1}{2}}}\,dw \\ &= \frac{\pi\,e^{-u}\,u^{2n-2}\,(2u-2n-1)}{2\,.\,(2n-1)\,!}\end{aligned}\right\} \; . \; . \; . \quad (87)$$

Hence $\displaystyle\int_0^\infty \phi\,(w)\sin uw\,dw$

$$= \frac{\pi e^{-u}}{b^2}\sum_{n=1}^{\infty}\left[\frac{n\,\{(2n-1)\,\mathrm{F}_{2n-1} + 2\mathrm{G}_{2n+1}\}\,u^{2n}}{(2n)\,!} + \frac{(n-1)\,(2n-1)\,\mathrm{G}_{2n-1}}{(2n-1)\,!}(2u-2n+1)u^{2n-2}\right]$$

$$= \frac{\pi e^{-u}}{2b^2}\sum_{n=1}^{\infty}\left[\frac{\mathrm{F}_{2n-1} - \mathrm{G}_{2n+1}}{(2n-2)\,!}\,u^{2n} + \frac{2\mathrm{G}_{2n+1}\,u^{2n+1}}{(2n-1)\,!}\right]. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (88)$$

Similarly $\displaystyle\int_0^\infty \psi\,(w)\sin uw\,dw$

$$= -\frac{\pi e^{-u}}{2b^2}\sum_{n=1}^{\infty}\left[\frac{(2n-1)\,\mathrm{F}_{2n-1} - (2n+1)\,\mathrm{G}_{2n+1}}{(2n-1)\,!}\,u^{2n} + \frac{2\mathrm{G}_{2n+1}\,u^{2n+1}}{(2n-1)\,!}\right]. \quad . \; . \; . \; . \; . \; (89)$$

Now write

$$\left.\begin{aligned}\Phi &= \frac{b^2}{\pi}\int_0^\infty \phi\,(w)\sin uw\,dw \\ \Psi &= \frac{b^2}{\pi}\int_0^\infty \psi\,(w)\cos uw\,dw\end{aligned}\right\}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (90)$$

Then

$$\chi_{2r+1} = 4\int_0^\infty \frac{\eta\mathrm{S}\sin u\xi}{u\Sigma}\,(s\,\Phi - c\,\Psi)\,du$$

$$- 4\int_0^\infty \frac{\mathrm{C}\sin u\xi}{u^2\Sigma}\,(\overline{s+uc}\;\Phi - us\,\Psi)\,du.$$

* " A," pp. 56 and 57. The change to half-integral indices is easily seen to be justifiable. The misprinted denominator of the integrand in equation (21) should be corrected to $(1 + w^2)^n$.

Using the expansions quoted in (11),

$$\chi_{2r+1} = 2\int_0^\infty \frac{s\Phi - c\Psi}{\Sigma} \sum_{n=0}^\infty \left[\frac{u^{2n-1}\,\rho^{2n+1}}{(2n)\,!} + \frac{u^{2n+1}\,\rho^{2n+3}}{(2n+2)\,!}\right] \sin\,(2n+1)\,\theta\,du$$

$$- 4\int_0^\infty \frac{(s+uc)\,\Phi - us\,\Psi}{\Sigma} \sum_{n=0}^\infty \frac{u^{2n-1}\,\rho^{2n+1}}{(2n+1)\,!} \sin\,(2n+1)\,\theta\,du$$

$$= \sum_{n=0}^\infty \left\{P^{(r)}_{2n+1} + Q^{(r)}_{2n+1}\,\rho^2\right\} \rho^{2n+1} \sin\,(2n+1)\,\theta, \quad \ldots \ldots \quad (91)^*$$

where

$$\left.\begin{aligned} P^{(r)}_{2n+1} &= 2\int_0^\infty \frac{s\Phi - c\Psi}{\Sigma}\,\frac{u^{2n-1}}{(2n)\,!}\,du - 4\int_0^\infty \frac{(s+uc)\,\Phi - us\,\Psi}{\Sigma}\,\frac{u^{2n-1}}{(2n+1)\,!}\,du \\ Q^{(r)}_{2n+1} &= 2\int_0^\infty \frac{s\Phi - c\Psi}{\Sigma}\,\frac{u^{2n+1}}{(2n+2)\,!}\,du \end{aligned}\right\} \ldots (92)$$

Now

$$2\,(s\,\Phi - c\Psi) = e^u\,(\Phi - \Psi) - e^{-u}\,(\Phi + \Psi),$$

$$2\,\{(s+uc)\;\Phi - us\Psi\} = e^u\,\{\Phi + u\,(\Phi - \Psi)\} + e^{-u}\,\{u\,(\Phi + \Psi) - \Phi\},$$

and from (88), (89) and (90)

$$\Phi + \Psi = e^{-u} \sum_{p=1}^\infty \frac{G_{2p+1}}{(2p-1)\,!}\,u^{2p}$$

$$\Phi - \Psi = e^{-u} \sum_{p=1}^\infty \left\{\frac{F_{2p-1}}{(2p-2)\,!}\,u^{2p} - \frac{2pG_{2p+1}}{(2p-1)\,!}\,u^{2p} + \frac{2G_{2p+1}}{(2p-1)\,!}\,u^{2p+1}\right\}.$$

Hence

$$2\int_0^\infty \frac{s\Phi - c\Psi}{\Sigma}\cdot\frac{u^{2n-1}}{(2n)\,!}\,du$$

$$= \sum_{p=1}^\infty \frac{I_{2n+2p-1}}{(2n)\,!\,(2p-2)\,!}\,F_{2p-1} - \sum_{p=1}^\infty \frac{1}{(2n)\,!\,(2p-1)\,!}\,\{2pI_{2n+2p-1} - 2I_{2n+2p} + J_{2n+2p-1}\}\,G_{2p+1}.$$

$$4\int_0^\infty \frac{(s+uc)\,\Phi - us\Psi}{\Sigma}\,\frac{u^{2n-1}}{(2n+1)\,!}\,du$$

$$= \frac{1}{(2n+1)\,!}\int_0^\infty \sum_{p=1}^\infty \left[\frac{F_{2p-1} - G_{2p+1}}{(2p-2)\,!}\,u^{2p} + \frac{2G_{2p+1}u^{2p+1}}{(2p-1)\,!} + \frac{2F_{2p-1}}{(2p-2)\,!}\,u^{2p+1} - \frac{4pG_{2p+1}}{(2p-1)\,!}\,u^{2p+1} + \frac{4G_{2p+1}}{(2p-1)\,!}\,u^{2p+2} - e^{-2u}\,\frac{F_{2p-1} - G_{2p+1}}{(2p-2)\,!}\,u^{2p}\right] u^{2n-1}\,du.$$

$$= \frac{1}{(2n+1)\,!} \sum_{p=1}^\infty \left[\frac{1}{(2p-2)\,!}\,\{I_{2p+2n-1} + 2I_{2p+2n} - J_{2p+2n-1}\}\,F_{2p-1} + \frac{1}{(2p-1)\,!}\,\{-(2p-1)\,I_{2p+2n-1} - 2\,(2p-1)\,I_{2p+2n} - 4I_{2p+2n+1} + (2p-1)\,J_{2p+2n-1}\}\,G_{2p+1}\right].$$

* A multiple of $\rho \sin \theta$ has been added; such a term is trivial as it adds nothing to the stresses.

Combining these results, we have

$$P^{(r)}{}_{2n+1} = \sum_{p=0}^{\infty} \left({}^{2n+1}\alpha_{2p+1} F^{(r)}{}_{2p+1} + {}^{2n+1}\beta_{2p+1} G^{(r)}{}_{2p+1}\right), \qquad (93)$$

where

$$ {}^{2n+1}\beta_1 = 0 \qquad (94)$$

and the other coefficients fall within the definitions (67).

$Q^{(r)}{}_{2n+1}$ is identical with the first integral in $P^{(r)}{}_{2n+1}$, except that n must be increased by 2. Hence

$$Q^{(r)}{}_{2n+1} = \sum_{p=0}^{\infty} \left({}^{2n+1}\gamma_{2p+1} F_{2p+1} + {}^{2n+1}\delta_{2p+1} G_{2p+1}\right) \qquad (95)$$

where

$$ {}^{2n+1}\delta_1 = 0 \qquad (96)$$

and the other coefficients are defined as in (67). Instead of making ${}^{2n+1}\beta_1 = {}^{2n+1}\delta_1 = 0$, we may omit the trivial G_1 term entirely, but the method here adopted results in a rather more compact notation.

The coefficients in (93) and (95) are given in Tables VII to X.

TABLE VII.—Values of $-\,{}^{n}\alpha_p$ (odd suffixes).

	$n = 3.$	$n = 5.$	$n = 7.$	$n = 9.$	$n = 11.$	$n = 13.$
$p = 1$	0·22942	0·062857	0·015900	0·003947	0·000981	0·000245
$p = 3$	1·2070	0·65309	0·28143	0·10751	0·038104	0·012820
$p = 5$	1·6205	1·4727	0·96643	0·52370	0·24995	0·10895
$p = 7$	1·3070	1·8029	1·6755	1·2219	0·75536	0·41400
$p = 9$	0·80458	1·5706	1·9636	1·8547	1·4416	0·97032
$p = 11$	0·41878	1·0996	1·7805	2·1144	2·0183	1·6366
$p = 13$	0·19437	0·66092	1·3455	1·9622	2·2565	2·1697

TABLE VIII.—Values of $-\,{}^{n}\beta_p$ (odd suffixes).

	$n = 1.$	$n = 3.$	$n = 5.$	$n = 7.$	$n = 9.$	$n = 11.$	$n = 13.$
$p = 3$	1·5518	1·0857	0·46578	0·16422	0·052844	0·016133	0·004763
$p = 5$	0·67181	1·5525	1·3623	0·81992	0·40289	0·17457	0·069435
$p = 7$	0·23439	1·1495	1·7218	1·5789	1·0997	0·64154	0·33051
$p = 9$	0·073020	0·63412	1·4504	1·8852	1·7675	1·3355	0·86496
$p = 11$	0·021875	0·29577	0·96014	1·6802	2·0404	1·9381	1·5418
$p = 13$	0·006395	0·12383	0·54159	1·2276	1·8741	2·1865	2·0953

TABLE IX.—Values of ${}^{n}\gamma_{p}$ (odd suffixes).

	$n = 1.$	$n = 3.$	$n = 5.$	$n = 7.$	$n = 9.$	$n = 11.$	$n = 13.$
$p = 1$	0·31039	0·14461	0·053197	0·017419	0·005356	0·001586	0·000458
$p = 3$	0·86765	0·79796	0·48773	0·24102	0·10465	0·041646	0·015563
$p = 5$	0·79796	1·2193	1·1248	0·78487	0·45811	0·23604	0·11089
$p = 7$	0·48773	1·1248	1·4651	1·3743	1·0386	0·67275	0·38813
$p = 9$	0·24102	0·78487	1·3743	1·6691	1·5858	1·2614	0·87675
$p = 11$	0·10465	0·45811	1·0386	1·5858	1·8501	1·7730	1·4612
$p = 13$	0·041646	0·23604	0·67275	1·2614	1·7730	2·0148	1·9428

TABLE X.—Values of ${}^{n}\delta_{p}$ (odd suffixes).

	$n = 1.$	$n = 3.$	$n = 5.$	$n = 7.$	$n = 9.$	$n = 11.$	$n = 13.$
$p = 3$	0·68826	0·60350	0·32409	0·14003	0·053639	0·019035	0·006408
$p = 5$	0·62857	1·0885	0·98182	0·64390	0·34901	0·16661	0·072629
$p = 7$	0·33391	0·98500	1·3530	1·2566	0·91637	0·56651	0·31050
$p = 9$	0·14208	0·64507	1·2569	1·5710	1·4837	1·1533	0·77625
$p = 11$	0·053951	0·34928	0·91647	1·4837	1·7620	1·6819	1·3638
$p = 13$	0·019077	0·16665	0·56652	1·1533	1·6819	1·9342	1·8598

The inverse transformation from a stress-function with an odd suffix to the succeeding one with an even suffix is now made as follows. The stresses produced by χ_{2r+1} are

$$\left.\begin{aligned}\widehat{rr} &= -\frac{2}{b^2}\sum_{n=0}^{\infty}\left[n(2n+1)\,P^{(r)}{}_{2n+1} + (n+1)(2n-1)\,Q^{(r)}{}_{2n+1}\rho^2\right]\rho^{2n-1}\sin(2n+1)\,\theta \\ \widehat{\theta\theta} &= \frac{2}{b^2}\sum_{n=0}^{\infty}\left[n(2n+1)\,P^{(r)}{}_{2n+1} + (n+1)(2n+3)\,Q^{(r)}{}_{2n+1}\rho^2\right]\rho^{2n-1}\sin(2n+1)\,\theta \\ \widehat{r\theta} &= -\frac{2}{b^2}\sum_{n=0}^{\infty}\left[n(2n+1)\,P^{(r)}{}_{2n+1} + (n+1)(2n+1)\,Q^{(r)}{}_{2n+1}\rho^2\right]\rho^{2n-1}\cos(2n+1)\,\theta\end{aligned}\right\} . \qquad (97)$$

Those due to χ_{2r+2} will be identical with (80), except that the upper suffix will be $(r+1)$. If they cancel $\widehat{rr}$ and $\widehat{r\theta}$ when $\rho = \lambda$, we have

$$\frac{(n+1)(2n+1)\,F^{(r+1)}{}_{2n+1}}{\lambda^{2n+3}} + \frac{n(2n+3)\,G^{(r+1)}{}_{2n+1}}{\lambda^{2n+1}} + n(2n+1)\,P^{(r)}{}_{2n+1}\lambda^{2n-1} + (n+1)(2n-1)\,Q^{(r)}{}_{2n+1}\lambda^{2n+1} = 0,$$

$$\frac{(n+1)(2n+1)\,F^{(r+1)}{}_{2n+1}}{\lambda^{2n+3}} + \frac{n(2n+1)\,G^{(r+1)}{}_{2n+1}}{\lambda^{2n+1}} - n(2n+1)\,P^{(r)}{}_{2n+1}\lambda^{2n-1} - (n+1)(2n+1)\,Q^{(r)}{}_{2n+1}\lambda^{2n+1} = 0,$$

whence

$$\left.\begin{aligned}F^{(r+1)}{}_{2n+1} &= \lambda^{4n+2}\left\{2nP^{(r)}{}_{2n+1} + (2n+1)\,\lambda^2Q^{(r)}{}_{2n+1}\right\} \\ G^{(r+1)}{}_{2n+1} &= -\lambda^{4n}\left\{(2n+1)\,P^{(r)}{}_{2n+1} + 2(n+1)\,\lambda^2Q^{(r)}{}_{2n+1}\right\}\end{aligned}\right\} \qquad (98)$$

This completes the cycle, giving χ_{2r+2} in the same from as χ_{2r} and all the terms in the series for χ''' may be determined, each from the preceding, when χ_0 is known.

Alternatively, the even-suffixed terms may be determined directly from each other without the intervention of those with odd suffixes. The direct transformation, which follows at once from (93), (95) and (98) is

$$\left.\begin{aligned} \mathrm{F}^{(r+1)}{}_{2n+1} &= \sum_{p=0}^{\infty} \{{}^{2n+1}h_{2p+1}\,\mathrm{F}^{(r)}{}_{2p+1} + {}^{2n+1}i_{2p+1}\,\mathrm{G}^{(r)}{}_{2p+1}\} \\ \mathrm{G}^{(r+1)}{}_{2n+1} &= \sum_{p=0}^{\infty} \{{}^{2n+1}j_{2p+1}\,\mathrm{F}^{(r)}{}_{2p+1} + {}^{2n+1}k_{2p+1}\,\mathrm{G}^{(r)}{}_{2p+1}\} \end{aligned}\right\}, \quad \ldots\ldots \quad (99)$$

where the new coefficients are now defined as in (70). The successive even stress-functions being deduced from these equations, we then write

$$\left.\begin{aligned} f_{2n+1} &= \sum_r \mathrm{F}^{(r)}{}_{2n+1}, \quad g_{2n+1} = \sum_r \mathrm{G}^{(r)}{}_{2n+1} \\ p_{2n+1} &= \sum_r \mathrm{P}^{(r)}{}_{2n+1}, \quad q_{2n+1} = \sum_r \mathrm{Q}^{(r)}{}_{2n+1} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (100)^*$$

The coefficients p_{2n+1}, q_{2n+1} are given, using (93) and (95) by

$$\left.\begin{aligned} p_{2n+1} &= \sum_{p=0}^{\infty} ({}^{2n+1}\alpha_{2p+1} f_{2p+1} + {}^{2n+1}\beta_{2p+1}\, g_{2p+1}) \\ q_{2n+1} &= \sum_{p=0}^{\infty} ({}^{2n+1}\gamma_{2p+1} f_{2p+1} + {}^{2n+1}\delta_{2p+1}\, g_{2p+1}) \end{aligned}\right\} \quad \ldots\ldots \quad (101)$$

and the final stress-function is

$$\chi''' = \sum_{n=0}^{\infty} \left\{\frac{f_{2n+1}}{\rho^{2n+1}} + \frac{g_{2n+1}}{\rho^{2n-1}} + (p_{2n+1} + q_{2n+1}\rho^2)\,\rho^{2n+1}\right\} \sin(2n+1)\,\theta. \quad . \quad . \quad (102)$$

This gives the following values for the components of stress

$$\left.\begin{aligned} \widehat{rr} &= -\frac{2}{b^2}\sum_{n=0}^{\infty}\left[\frac{(n+1)(2n+1) f_{2n+1}}{\rho^{2n+3}} + \frac{n(2n+3)\, g_{2n+1}}{\rho^{2n+1}}\right. \\ &\quad \left. + \{n(2n+1)\,p_{2n+1} + (n+1)(2n-1)\,q_{2n+1}\rho^2\}\,\rho^{2n-1}\right] \sin(2n+1)\,\theta \\ \widehat{\theta\theta} &= \frac{2}{b^2}\sum_{n=0}^{\infty}\left[\frac{(n+1)(2n+1) f_{2n+1}}{\rho^{2n+3}} + \frac{n(2n-1)\, g_{2n+1}}{\rho^{2n+1}}\right. \\ &\quad \left. + \{n(2n+1)\,p_{2n+1} + (n+1)(2n+3)\,q_{2n+1}\rho^2\}\,\rho^{2n-1}\right] \sin(2n+1)\,\theta \\ \widehat{r\theta} &= \frac{2}{b^2}\sum_{n=0}^{\infty}\left[(n+1)(2n+1)\left\{\frac{f_{2n+1}}{\rho^{2n+3}} - q_{2n+1}\rho^{2n+1}\right\}\right. \\ &\quad \left. + n(2n+1)\left\{\frac{g_{2n+1}}{\rho^{2n+1}} - p_{2n+1}\,\rho^{2n-1}\right\}\right] \cos(2n+1)\,\theta \end{aligned}\right\}. \quad (103)$$

The values of the coefficients in the direct transformation (99), for $\lambda = 0\cdot2$, $0\cdot3$, $0\cdot4$, $0\cdot5$ have been computed, but our tables of them are omitted.

* See footnote to equations (72), p. 179.

(iii) χ''' *even in x and odd in y.*

In this and the following section the analysis, being similar to that already given, will be abbreviated as much as possible. There are, however, a few new features which will call for special comment.

χ''' is to satisfy the conditions :—

(i) $\widehat{rr}, \widehat{r\theta}, \widehat{\theta\theta}$ all $\rightarrow 0$ when $x \rightarrow \infty$;

(ii) $\widehat{yy} = \widehat{xy} = 0$ when $\eta = \pm 1$;

(iii) $\widehat{rr} = f_5(\theta), \widehat{r\theta} = f_6(\theta)$ when $\rho = \lambda, f_5(\theta), f_6(\theta)$ being functions expressible in the form

$$\left.\begin{aligned} f_5(\theta) &= \sum_{n=0}^{\infty} S_{2n+1} \cos(2n+1)\theta \\ f_6(\theta) &= \sum_{n=0}^{\infty} T_{2n+1} \sin(2n+1)\theta \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (104)$$

To satisfy these we begin with

$$\chi_0 = \sum_{n=0}^{\infty} \left\{\frac{D^{(0)}_{2n+1}}{\rho^{2n+1}} + \frac{E^{(0)}_{2n+1}}{\rho^{2n-1}}\right\} \cos(2n+1)\theta, \quad \ldots \ldots \quad (105)$$

where

$$\left.\begin{aligned} D^{(0)}_{2n+1} &= \frac{b^2\lambda^{2n+3}\{(2n+1)S_{2n+1} - (2n+3)T_{2n+1}\}}{4(n+1)(2n+1)} \\ E^{(0)}_{2n+1} &= \frac{b^2\lambda^{2n+1}(T_{2n+1} - S_{2n+1})}{4n} \end{aligned}\right\} \quad \ldots \quad (106)$$

Next we write

$$\chi''' = \chi_0 + \chi_1 + \chi_2 + \chi_3 + \ldots, \quad \ldots \ldots \ldots \quad (58\ bis)$$

when $\chi_1, \chi_2, \ldots$ are defined in succession as before. We have now to find formulæ expressing the coefficients of χ_{2r+2} in terms of those of χ_{2r} and χ_{2r+1}. Let

$$\chi_{2r} = \sum_{n=0}^{\infty} \left\{\frac{D^{(r)}_{2n+1}}{\rho^{2n+1}} + \frac{E^{(r)}_{2n+1}}{\rho^{2n-1}}\right\} \cos(2n+1)\theta. \quad \ldots \ldots \quad (107)$$

Then, the upper suffixes being temporarily omitted, the stress-components are

$$\left.\begin{aligned} \widehat{rr} &= -\frac{2}{b^2}\sum_{n=0}^{\infty}\left\{\frac{(n+1)(2n+1)D_{2n+1}}{\rho^{2n+3}} + \frac{n(2n+3)E_{2n+1}}{\rho^{2n+1}}\right\}\cos(2n+1)\theta \\ \widehat{r\theta} &= -\frac{2}{b^2}\sum_{n=0}^{\infty}\left\{\frac{(n+1)(2n+1)D_{2n+1}}{\rho^{2n+3}} + \frac{n(2n+1)E_{2n+1}}{\rho^{2n+1}}\right\}\sin(2n+1)\theta \\ \widehat{\theta\theta} &= \frac{2}{b^2}\sum_{n=0}^{\infty}\left\{\frac{(n+1)(2n+1)D_{2n+1}}{\rho^{2n+3}} + \frac{n(2n-1)E_{2n+1}}{\rho^{2n+1}}\right\}\cos(2n+1)\theta \end{aligned}\right\}. \quad (108)$$

From these, in the same way as before, the Cartesian stress-components are found to be

$$\left.\begin{aligned}\widehat{xx} &= \frac{2}{b^2}\sum_{n=0}^{\infty}\left[\frac{(n+1)\{(2n+1)\,\mathrm{D}_{2n+1}-2\mathrm{E}_{2n+3}\}}{\rho^{2n+3}}+\frac{n\,(2n+1)\,\mathrm{E}_{2n+1}}{\rho^{2n+1}}\right]\cos\,(2n+3)\,\theta \\ \widehat{yy} &= -\frac{2}{b^2}\sum_{n=0}^{\infty}\left[\frac{(n+1)\{(2n+1)\,\mathrm{D}_{2n+1}+2\mathrm{E}_{2n+3}\}}{\rho^{2n+3}}+\frac{n\,(2n+1)\,\mathrm{E}_{2n+1}}{\rho^{2n+1}}\right]\cos\,(2n+3)\,\theta \\ \widehat{xy} &= -\frac{2}{b^2}\sum_{n=0}^{\infty}\left[\frac{(n+1)\,(2n+1)\,\mathrm{D}_{2n+1}}{\rho^{2n+3}}+\frac{n\,(2n+1)\,\mathrm{E}_{2n+1}}{\rho^{2n+1}}\right]\sin\,(2n+3)\,\theta\end{aligned}\right\} \quad \ldots \ldots (109)$$

χ_{2r+1}, if it is to cancel the tractions due to χ_{2r} on the lines $\eta = \pm 1$, must itself produce tractions

$$\widehat{yy} = \phi\,(\xi), \quad \widehat{xy} = \psi\,(\xi), \quad \text{when } \eta = 1,$$

$$\widehat{yy} = -\phi\,(\xi), \quad \widehat{xy} = \psi\,(\xi), \quad \text{when } \eta = -1,$$

where

$$\left.\begin{aligned}\phi\,(\xi) &= \frac{2}{b^2}\sum_{n=0}^{\infty}\left[\frac{(n+1)\{(2n+1)\,\mathrm{D}_{2n+1}+2\mathrm{E}_{2n+3}\}}{(1+\xi^2)^{\frac{2n+3}{2}}}+\frac{n\,(2n+1)\,\mathrm{E}_{2n+1}}{(1+\xi^2)^{\frac{2n+1}{2}}}\right]\cos\,(2n+3)\,\theta \\ \psi\,(\xi) &= \frac{2}{b^2}\sum_{n=0}^{\infty}\left[\frac{(n+1)\,(2n+1)\,\mathrm{D}_{2n+1}}{(1+\xi^2)^{\frac{2n+3}{2}}}+\frac{n\,(2n+1)\,\mathrm{E}_{2n+1}}{(1+\xi^2)^{\frac{2n+1}{2}}}\right]\sin\,(2n+3)\,\theta\end{aligned}\right\}, \quad \ldots \ldots (110)$$

and

$$\tan\theta = \xi. \quad \ldots \ldots (111)$$

Hence ("B," p. 93)

$$\chi_{2r+1} = \chi'_{2r+1} + \chi''_{2r+1} \quad \ldots \ldots (112)$$

where

$$\left.\begin{aligned}\chi'_{2r+1} &= \frac{4b^2}{\pi}\int_0^{\infty}\frac{u\eta c\mathrm{C}-(c+us)\,\mathrm{S}}{u^2\Sigma'}\cos u\xi\,du\int_0^{\infty}\phi\,(w)\cos uw\,dw \\ \chi''_{2r+1} &= \frac{4b^2}{\pi}\int_0^{\infty}\frac{\eta s\mathrm{C}-c\mathrm{S}}{u\Sigma'}\cos u\xi\,du\int_0^{\infty}\psi\,(w)\sin uw\,dw\end{aligned}\right\}. \quad (113)$$

Using the results of (87), we obtain

$$\left.\begin{aligned}\int_0^{\infty}\phi\,(w)\cos uw\,dw &= \frac{\pi u}{b^2}\,\Phi \\ \int_0^{\infty}\psi\,(w)\sin uw\,dw &= \frac{\pi u}{b^2}\,\Psi\end{aligned}\right\}, \quad \ldots \ldots (114)$$

where

$$\left.\begin{aligned}\Phi &= \tfrac{1}{2}e^{-u}\sum_{n=0}^{\infty}\frac{1}{(2n+1)\,!}\{(2n+1)\,(\mathrm{D}_{2n+1}-\mathrm{E}_{2n+3})\,u^{2n+1}+2\mathrm{E}_{2n+3}u^{2n+2}\} \\ \Psi &= \tfrac{1}{2}e^{-u}\sum_{n=0}^{\infty}\frac{1}{(2n+1)\,!}[\{(2n+1)\,\mathrm{D}_{2n+1}-(2n+3)\,\mathrm{E}_{2n+3}\}\,u^{2n+1}+2\mathrm{E}_{2n+3}u^{2n+2}]\end{aligned}\right\}. \quad (115)$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

It is not obvious that the integrals in (113) converge, but this will be assumed for the present. After χ_{2r+1} has been expanded, the terms that are not obviously convergent will be examined separately. We have now

$$\chi_{2r+1} = 4\int_0^\infty \frac{\eta C \cos u\xi}{\Sigma'}(c\,\Phi + s\Psi)\,du - 4\int_0^\infty \frac{S\cos u\xi}{u\Sigma'}\{(c+us)\ \Phi + uc\Psi\}\,du.$$

In this we substitute the expansions of $\eta C \cos u\xi$ and $S \cos u\xi$ given in (24). Then

$$\chi_{2r+1} = \sum_{n=0}^{\infty}\{L^{(r)}{}_{2n+1} + M^{(r)}{}_{2n+1}\,\rho^2\}\rho^{2n+1}\cos(2n+1)\,\theta \quad . \quad . \quad . \quad . \quad . \quad (116)$$

the coefficients being

$$\left.\begin{aligned} L_1^{(r)} &= 4\int_0^\infty\{s(\Psi - u\Phi) - uc\Psi\}\frac{du}{\Sigma'},\\ L^{(r)}{}_{2n+1} &= \frac{2}{(2n+1)\,!}\int_0^\infty [\{(2n+1)\,\Phi - 2u\Psi\}\,c + \{(2n+1)\Psi - 2u\Phi\}\,s]\,\frac{u^{2n}}{\Sigma'}\,du\\ M^{(r)}{}_{2n+1} &= \frac{2}{(2n+2)\,!}\int_0^\infty (c\Phi + s\Psi)\,\frac{u^{2n+2}}{\Sigma'}\,du \end{aligned}\right\},\ (117)$$

the second equation applying for $n > 0$ and the third for $n \geq 0$.

Except when $n = 0$, the reduction of these coefficients follows the same lines as in the previous cases. We find

$$\left.\begin{aligned} L^{(r)}{}_{2n+1} &= \sum_{p=0}^{\infty}\{{}^{2n+1}\kappa_{2p+1}\,D^{(r)}{}_{2p+1} + {}^{2n+1}\mu_{2p+1}\,E^{(r)}{}_{2p+1}\}\\ M^{(r)}{}_{2n+1} &= \sum_{p=0}^{\infty}\{{}^{2n+1}\nu_{2p+1}\,D^{(r)}{}_{2p+1} + {}^{2n+1}\varpi_{2p+1}\,E^{(r)}{}_{2p+1}\} \end{aligned}\right\}, \quad . \quad . \quad . \quad (118)$$

where, if $n > 1$, $p > 0$,

$$\left.\begin{aligned} {}^{n}\kappa_p &= \frac{1}{n\,!\,(p-1)\,!}\{(n-1)\,I'_{n+p-1} - 2I'_{n+p} - J'_{n+p-1}\}\\ {}^{n}\mu_p &= \frac{1}{n\,!\,(p-2)\,!}\{2\,(n+p-2)\,I'_{n+p-2} - (np-n-p+2)\,I'_{n+p-3}\\ &\qquad - 4I'_{n+p-1} + (n+p-2)\,J'_{n+p-3}\}\\ {}^{n}\nu_p &= \frac{I'_{n+p+1}}{(n+1)\,!\,(p-1)\,!}\\ {}^{n}\varpi_p &= -\frac{1}{(n+1)!\,(p-2)!}\{(p-1)\,I'_{n+p-1} - 2I'_{n+p} - J'_{n+p-1}\} \end{aligned}\right\} \quad . \quad . \quad (119)$$

These coefficients are very similar in form to those defined in (67); they may be derived from them if every I_s is replaced by I'_s and every J_s by $-J'_s$.

In the above equations $L_1^{(r)}$ and $M_1^{(r)}$ were excluded from consideration. They will now be discussed. Taking $L_1^{(r)}$ first, we have

$$\Phi = \tfrac{1}{2}(D_1 - E_3)\,u + O(u^2)$$
$$\Psi = \tfrac{1}{2}(D_1 - 3E_3)\,u + O(u^2)$$

so that $s(\Psi - u\Phi) - uc\Psi = O(u^3)$ and the integral converges. Putting in the full values of Φ and Ψ we have, after some reduction,

$$4\{s(\Psi - u\Phi) - uc\Psi\}$$
$$= \sum_{p=0}^{\infty} \frac{1}{(2p)!}\{(1 - e^{-2u})\,u^{2p+1} - 2u^{2p+2}\}\,D^{(r)}{}_{2p+1}$$
$$+ \sum_{p=0}^{\infty} \frac{1}{(2p+1)!}[(2p+3)\{2u^{2p+2} - (1 - e^{-2u})\,u^{2p+1}\} - 4u^{2p+3}]\,E^{(r)}{}_{2p+3}.$$

Hence

$$L_1^{(r)} = \sum_{p=0}^{\infty} \{{}^1\kappa_{2p+1}\,D^{(r)}{}_{2p+1} + {}^1\mu_{2p+1}\,E^{(r)}{}_{2p+1}\} \quad . \; . \; . \; . \; . \; . \; . \quad (120)$$

where, if $p > 0$,

$$\left.\begin{aligned} {}^1\kappa_{2p+1} &= \frac{1}{(2p)!}\{I'_{2p+1} - 2I'_{2p+2} - J'_{2p+1}\} \\ {}^1\mu_{2p+1} &= \frac{1}{(2p-1)!}\{2p+1)(2I'_{2p} - I'_{2p-1} + J'_{2p-1}) - 4I'_{2p+1}\} \end{aligned}\right\} \quad . \; . \; . \quad (121)$$

It is also obvious that

$${}^1\mu_1 = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (122)$$

Finally

$$\begin{aligned} {}^1\kappa_1 &= \int_0^{\infty} \frac{(1 - e^{-2u})\,u - 2u^2}{\Sigma'}\,du \\ &= \int_0^{\infty} \left\{\frac{u}{\Sigma'} - \frac{3(1+u)}{4u^2}\,e^{-u}\right\} du - 2\int_0^{\infty} \left\{\frac{u^2}{\Sigma'} - \frac{3}{4u}\,e^{-u}\right\} du \\ &\qquad - \int_0^{\infty} \left\{\frac{u}{\Sigma'}\,e^{-2u} - \frac{3(1-u)}{4u^2}\,e^{-u}\right\} du \end{aligned}$$

or

$${}^1\kappa_1 = I'_1 - 2I'_2 - J'_1, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (123)$$

using the definitions given in (28).

Examination of $M_1^{(r)}$ shows that the integral converges and that

$$M_1^{(r)} = \sum_{p=0}^{\infty} \{{}^1\nu_{2p+1}\,D^{(r)}{}_{2p+1} + {}^1\varpi_{2p+1}\,E^{(r)}{}_{2p+1}\} \quad . \; . \; . \; . \; . \; . \; . \quad (124)$$

where all the coefficients conform to (119) except that

$$ {}^{1}\varpi_{1} = 0. \qquad (125) $$

When $(n + p)$ is large the coefficients may be calculated from asymptotic formulæ identical with those given for the previous set of coefficients. We have, in fact,

$$ \left.\begin{aligned} {}^{n}\kappa_{p} &\sim -\frac{(p+1)(n+p-1)!}{2^{n+p-1}n!\,(p-1)!} \sim {}^{n}\alpha_{p} \\ {}^{n}\mu_{p} &\sim -\frac{p\,(n+p-3)!}{2^{n+p-3}(n-1)!\,(p-2)!} \sim {}^{n}\beta_{p} \\ {}^{n}\nu_{p} &\sim \frac{(n+p+1)!}{2^{n+p+1}(n+1)!\,(p-1)!} \sim {}^{n}\gamma_{p} \\ {}^{n}\varpi_{p} &\sim \frac{(n+p-1)!}{2^{n+p-1}\,n!\,(p-2)!} \sim {}^{n}\delta_{p} \end{aligned}\right\} \qquad (126) $$

Of the values given in Tables XI to XIV, those for which $n + p > 22$ have been calculated from (126). For the others the general formulæ (119), (121) and (123) have been used.

TABLE XI.—Values of $- {}^{n}\kappa_{p}$ (odd suffixes).

	$n = 3.$	$n = 5.$	$n = 7.$	$n = 9.$	$n = 11.$	$n = 13.$
$p = 1$	0·22456	0·058292	0·015111	0·003851	0·000971	0·000244
$p = 3$	1·3060	0·65833	0·28081	0·10730	0·038065	0·012815
$p = 5$	1·6656	1·4804	0·96704	0·52364	0·24992	0·10894
$p = 7$	1·3186	1·8064	1·6761	1·2219	0·75537	0·41400
$p = 9$	0·80678	1·5715	1·9639	1·8547	1·4416	0·97032
$p = 11$	0·41911	1·0998	1·7805	2·1144	2·0183	1·6366
$p = 13$	0·19442	0·66097	1·3455	1·9622	2·2565	2·1697

TABLE XII.—Values of $- {}^{n}\mu_{p}$ (odd suffixes).

	$n = 3.$	$n = 5.$	$n = 7.$	$n = 9.$	$n = 11.$	$n = 13.$
$p = 3$	1·2294	0·47199	0·16356	0·052605	0·016094	0·004758
$p = 5$	1·5733	1·3722	0·82081	0·40282	0·17455	0·069425
$p = 7$	1·1449	1·7237	1·5795	1·0998	0·64569	0·33047
$p = 9$	0·63127	1·4501	1·8853	1·7675	1·3354	0·86497
$p = 11$	0·29507	0·96000	1·6802	2·0402	1·9382	1·5417
$p = 13$	0·12372	0·54151	1·2275	1·8741	2·1864	2·0953

TABLE XIII.—Values of ${}^n\nu_p$ (odd suffixes).

	$n = 1.$	$n = 3.$	$n = 5.$	$n = 7.$	$n = 9.$	$n = 11.$	$n = 13.$
$p = 1$	0·76451	0·18076	0·056834	0·017769	0·005388	0·001588	0·000458
$p = 3$	1·0846	0·85251	0·49754	0·24244	0·10483	0·041667	0·015565
$p = 5$	0·85251	1·2438	1·1314	0·78621	0·45833	0·23607	0·11090
$p = 7$	0·49754	1·1314	1·4676	1·3750	1·0387	0·67277	0·38813
$p = 9$	0·24244	0·78621	1·3750	1·6693	1·5858	1·2614	0·87675
$p = 11$	0·10483	0·45833	1·0387	1·5858	1·8501	1·7730	1·4612
$p = 13$	0·041667	0·23607	0·67277	1·2614	1·7730	2·0148	1·9428

TABLE XIV.—Values of ${}^n\varpi_p$ (odd suffixes).

	$n = 1.$	$n = 3.$	$n = 5.$	$n = 7.$	$n = 9.$	$n = 11.$	$n = 13.$
$p = 3$	0·67368	0·65298	0·33311	0·14128	0·053785	0·019051	0·006409
$p = 5$	0·58292	1·0972	0·98692	0·64515	0·34923	0·16664	0·072634
$p = 7$	0·31733	0·98284	1·3538	1·2570	0·91650	0·56654	0·31050
$p = 9$	0·13866	0·64378	1·2567	1·5711	1·4838	1·1533	0·77625
$p = 11$	0·053420	0·34893	0·91639	1·4838	1·7620	1·6819	1·3638
$p = 13$	0·019006	0·16659	0·56651	1·1533	1·6819	1·9342	1·8598

To complete the formulæ of this section, the transformation from χ_{2r+1} to χ_{2r+2} must be found. The stresses due to χ_{2r+1} are

$$\left.\begin{aligned}\widehat{rr} &= -\frac{2}{b^2}\sum_{n=0}^{\infty}\{n(2n+1)\,L^{(r)}{}_{2n+1} + (n+1)(2n-1)\,M^{(r)}{}_{2n+1}\rho^2\}\,\rho^{2n-1}\cos(2n+1)\,\theta\\ \widehat{r\theta} &= \frac{2}{b^2}\sum_{n=0}^{\infty}\{n(2n+1)\,L^{(r)}{}_{2n+1} + (n+1)(2n+1)\,M^{(r)}{}_{2n+1}\rho^2\}\,\rho^{2n-1}\sin(2n+1)\,\theta\\ \widehat{\theta\theta} &= \frac{2}{b^2}\sum_{n=0}^{\infty}\{n(2n+1)\,L^{(r)}{}_{2n+1} + (n+1)(2n+3)\,M^{(r)}{}_{2n+1}\rho^2\}\,\rho^{2n-1}\cos(2n+1)\,\theta\end{aligned}\right\} \quad (127)$$

If now

$$\chi_{2r+2} = \sum_{n=0}^{\infty}\left\{\frac{D^{(r+1)}{}_{2n+1}}{\rho^{2n+1}} + \frac{E^{(r+1)}{}_{2n+1}}{\rho^{2n-1}}\right\}\cos(2n+1)\,\theta, \quad (128)$$

the stresses due to χ_{2r+2} are given by (108), except for the change of the upper suffix to $(r+1)$. The $\widehat{rr}$ and $\widehat{r\theta}$ components must cancel those from χ_{2r+1} when $\rho=\lambda$. Hence we find

$$\left.\begin{aligned} D^{(r)}_{2n+1} &= \lambda^{4n+2}\{2nL^{(r)}_{2n+1} + (2n+1)\lambda^2 M^{(r)}_{2n+1}\} \\ E^{(r)}_{2n+1} &= -\lambda^{4n}\{(2n+1)L^{(r)}_{2n+1} + 2(n+1)\lambda^2 M^{(r)}_{2n+1}\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (129)$$

From (118), (120), (124) and (129) we have the equations for the direct transformation from χ_{2n} to χ_{2n+2}. These are

$$\left.\begin{aligned} D^{(r+1)}_{2n+1} &= \sum_{p=0}^{\infty}\{{}^{2n+1}s_{2p+1}\,D^{(r)}_{2p+1} + {}^{2n+1}t_{2p+1}\,E^{(r)}_{2p+1}\} \\ E^{(r+1)}_{2n+1} &= \sum_{p=0}^{\infty}\{{}^{2n+1}u_{2p+1}\,D^{(r)}_{2p+1} + {}^{2n+1}v_{2p+1}\,E^{(r)}_{2p+1}\} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \qquad (130)$$

where

$$\left.\begin{aligned} {}^{n}s_p &= \lambda^{2n}\{(n-1)\,{}^{n}\kappa_p + n\lambda^2\,{}^{n}\nu_p\} \\ {}^{n}t_p &= \lambda^{2n}\{(n-1)\,{}^{n}\mu_p + n\lambda^2\,{}^{n}\varpi_p\} \\ {}^{n}u_p &= -\lambda^{2n-2}\{n\,{}^{n}\kappa_p + (n+1)\lambda^2\,{}^{n}\nu_p\} \\ {}^{n}v_p &= -\lambda^{2n-2}\{n\,{}^{n}\mu_p + (n+1)\lambda^2\,{}^{n}\varpi_p\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (131)$$

These definitions apply not only to the coefficients with odd suffixes used in (130), but also to those with even suffixes, which will be used in the next section.

As in the previous cases, when a sufficient number of coefficients have been found from (130), the solution may be completed by writing

$$\left.\begin{aligned} d_{2n+1} &= \sum_r D^{(r)}_{2n+1}, \qquad e_{2n+1} = \sum_r E^{(r)}_{2n+1}, \\ l_{2n+1} &= \sum_r L^{(r)}_{2n+1}, \qquad m_{2n+1} = \sum_r M^{(r)}_{2n+1} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (132)^*$$

and determining l_{2n+1}, m_{2n+1} from the equations

$$\left.\begin{aligned} l_{2n+1} &= \sum_{p=0}^{\infty}\{{}^{2n+1}\kappa_{2p+1}\,d_{2p+1} + {}^{2n+1}\mu_{2p+1}\,e_{2p+1}\} \\ m_{2n+1} &= \sum_{p=0}^{\infty}\{{}^{2n+1}\nu_{2p+1}\,d_{2p+1} + {}^{2n+1}\varpi_{2p+1}\,e_{2p+1}\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (133)$$

which follow at once from (118), (120), (124) and (132).

The final stress-function is now

$$\chi''' = \sum_{n=0}^{\infty}\left\{\frac{d_{2n+1}}{\rho^{2n+1}} + \frac{e_{2n+1}}{\rho^{2n-1}} + l_{2n+1}\,\rho^{2n+1} + m_{2n+1}\,\rho^{2n+3}\right\}\cos(2n+1)\theta \quad . \quad . \quad . \qquad (134)$$

* See footnote to equations (72), p. 179.

and the corresponding stress-components are

$$\left.\begin{aligned}\widehat{rr} &= -\frac{2}{b^2}\sum_{n=0}^{\infty}\left\{\frac{(n+1)(2n+1)\,d_{2n+1}}{\rho^{2n+3}}+\frac{n(2n+3)\,e_{2n+1}}{\rho^{2n+1}}\right.\\&\qquad\left.+\,n(2n+1)\,l_{2n+1}\,\rho^{2n-1}+(n+1)(2n-1)\,m_{2n+1}\,\rho^{2n+1}\right\}\cos(2n+1)\,\theta\\ \widehat{r\theta} &= \frac{2}{b^2}\sum_{n=0}^{\infty}\left\{(n+1)(2n+1)\left(l_{2n+1}\,\rho^{2n-1}-\frac{e_{2n+1}}{\rho^{2n+1}}\right)\right.\\&\qquad\left.+\,(n+1)(2n+3)\left(m_{2n+1}\,\rho^{2n+1}-\frac{d_{2n+1}}{\rho^{2n+3}}\right)\right\}\sin(2n+1)\,\theta\\ \widehat{\theta\theta} &= \frac{2}{b^2}\sum_{n=0}^{\infty}\left\{\frac{(n+1)(2n+1)\,d_{2n+1}}{\rho^{2n+3}}+\frac{n(2n-1)e_{2n+1}}{\rho^{2n+1}}+n(2n+1)\,l_{2n+1}\,\rho^{2n-1}\right.\\&\qquad\left.+\,(n+1)(2n+3)\,m_{2n+1}\,\rho^{2n+1}\right\}\cos(2n+1)\,\theta\end{aligned}\right\}\quad(135)$$

From these the Cartesian components $\widehat{xx}$, $\widehat{xy}$, $\widehat{yy}$ are readily deduced, using (81).

The values of the coefficients required in the direct transformation (130) have been tabulated, but our tables are omitted.

(iv) χ''' *odd in both x and y.*

Let the boundary conditions on the circle $\rho = \lambda$ now be

$$\widehat{rr} = f_7(\theta), \quad \widehat{r\theta} = f_8(\theta),$$

where these functions have expansions of the types

$$\left.\begin{aligned}f_7(\theta) &= \sum_{n=1}^{\infty} U_{2n}\sin 2n\theta\\ f_8(\theta) &= \sum_{n=1}^{\infty} V_{2n}\cos 2n\theta\end{aligned}\right\}\quad(136)$$

Then

$$\chi_0 = \sum_{n=1}^{\infty}\left(\frac{F^{(0)}_{2n}}{\rho^{2n}}+\frac{G^{(0)}_{2n}}{\rho^{2n-2}}\right)\sin 2n\,\theta,\quad(137)$$

where

$$\left.\begin{aligned}F^{(0)}_{2n} &= \frac{b^2\,\lambda^{2n+2}\,\{n\,U_{2n}+(n+1)\,V_{2n}\}}{2n\,(2n+1)}\\ G^{(0)}_{2n} &= \frac{b^2\,\lambda^{2n}\,(U_{2n}+V_{2n})}{2\,(2n-1)}\end{aligned}\right\}\quad(138)$$

Starting now with

$$\chi_{2r} = \sum_{n=1}^{\infty}\left(\frac{F^{(r)}_{2n}}{\rho^{2n}}+\frac{G^{(r)}_{2n}}{\rho^{2n-2}}\right)\sin 2n\,\theta.\quad(139)$$

we obtain the stresses

$$\left.\begin{aligned}\widehat{rr} &= -\frac{2}{b^2}\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)F^{(r)}_{2n}}{\rho^{2n+2}}+\frac{(n+1)(2n-1)G^{(r)}_{2n}}{\rho^{2n}}\right\}\sin 2n\theta \\ \widehat{r\theta} &= \frac{2}{b^2}\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)F^{(r)}_{2n}}{\rho^{2n+2}}+\frac{n(2n-1)G^{(r)}_{2n}}{\rho^{2n}}\right\}\cos 2n\theta \\ \widehat{\theta\theta} &= \frac{2}{b^2}\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)F^{(r)}_{2n}}{\rho^{2n+2}}+\frac{(n-1)(2n-1)G^{(r)}_{2n}}{\rho^{2n}}\right\}\sin 2n\theta\end{aligned}\right\} \quad \ldots\ldots \quad (140)$$

From these, using (81), we derive

$$\left.\begin{aligned}\widehat{xx} &= \frac{2}{b^2}\sum_{n=1}^{\infty}\left[\frac{(2n-1)\{(n-1)F^{(r)}_{2n-2}-G^{(r)}_{2n}\}}{\rho^{2n}}+\frac{(n-1)(2n-3)G^{(r)}_{2n-2}}{\rho^{2n-2}}\right]\sin 2n\theta \\ \widehat{yy} &= -\frac{2}{b^2}\sum_{n=1}^{\infty}\left[\frac{(2n-1)\{(n-1)F^{(r)}_{2n-2}+G^{(r)}_{2n}\}}{\rho^{2n}}+\frac{(n-1)(2n-3)G^{(r)}_{2n-2}}{\rho^{2n-2}}\right]\sin 2n\theta \\ \widehat{xy} &= \frac{2}{b^2}\sum_{n=1}^{\infty}\left[\frac{(n-1)(2n-1)F^{(r)}_{2n-2}}{\rho^{2n}}+\frac{(n-1)(2n-3)G^{(r)}_{2n-2}}{\rho^{2n-2}}\right]\cos 2n\theta\end{aligned}\right\} . \quad \ldots \quad (141)$$

To cancel $\widehat{yy}$ and $\widehat{xy}$ on the lines $\eta = \pm 1$ we now take ("B," p. 94)

$$\chi_{2r+1} = \chi'_{2r+1} + \chi''_{2r+1}, \quad \ldots\ldots\ldots\ldots \quad (142)$$

where

$$\left.\begin{aligned}\chi'_{2r+1} &= \frac{4b^2}{\pi}\int_0^{\infty}\frac{u\eta c\mathrm{C}-(c+us)\mathrm{S}}{u^2\Sigma'}\sin u\xi\, du\int_0^{\infty}\phi(w)\sin uw\, dw \\ \chi''_{2r+1} &= \frac{4b^2}{\pi}\int_0^{\infty}\frac{c\mathrm{S}-\eta s\mathrm{C}}{u\Sigma'}\sin u\xi\, du\int_0^{\infty}\psi(w)\cos uw\, dw\end{aligned}\right\} \quad \ldots \quad (143)$$

and

$$\left.\begin{aligned}\phi(w) &= \frac{2}{b^2}\sum_{n=1}^{\infty}\left[\frac{(2n-1)\{(n-1)F^{(r)}_{2n-2}+G^{(r)}_{2n}\}}{(1+w^2)^n}+\frac{(n-1)(2n-3)G^{(r)}_{2n-2}}{(1+w^2)^{n-1}}\right]\sin 2n\theta \\ \psi(w) &= -\frac{2}{b^2}\sum_{n=1}^{\infty}\left[\frac{(n-1)(2n-1)F^{(r)}_{2n-2}}{(1+w^2)^n}+\frac{(n-1)(2n-3)G^{(r)}_{2n-2}}{(1+w^2)^{n-1}}\right]\cos 2n\theta\end{aligned}\right\} . \quad \ldots \quad (144)$$

with

$$\tan\theta = w \quad \ldots\ldots\ldots\ldots\ldots \quad (145)$$

But we have ("A," pp. 56–57)

$$\left.\begin{aligned}\int_0^{\infty}\frac{\sin 2n\theta\sin uw}{(1+w^2)^n}\,dw &= \int_0^{\infty}\frac{\cos 2n\theta\cos uw}{(1+w^2)^n}\,dw = \frac{\pi e^{-u}u^{2n-1}}{2(2n-1)!} \\ \int_0^{\infty}\frac{\sin 2n\theta\sin uw}{(1+w^2)^{n-1}}\,dw &= \int_0^{\infty}\frac{\cos 2n\theta\cos uw}{(1+w^2)^{n-1}}\,dw = \frac{\pi e^{-u}u^{2n-3}(u-n+1)}{(2n-2)!}\end{aligned}\right\} . \quad . \quad (146)$$

Hence

$$\left.\begin{aligned}\int_0^\infty \phi(w)\sin uw\,dw &= \frac{\pi u}{b^2}\,\Phi\\ \int_0^\infty \psi(w)\cos uw\,dw &= \frac{\pi u}{b^2}\,\Psi\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (147)$$

where

$$\left.\begin{aligned}\Phi &= \tfrac{1}{2}e^{-u}\sum_{n=1}^{\infty}\left[\frac{u^{2n}}{(2n-1)!}\,F^{(r)}{}_{2n} + \frac{2\{u^{2n-1}-(n-1)\,u^{2n-2}\}}{(2n-2)!}\,G^{(r)}{}_{2n}\right]\\ \Psi &= -\tfrac{1}{2}e^{-u}\sum_{n=1}^{\infty}\left[\frac{u^{2n}}{(2n-1)!}\,F^{(r)}{}_{2n} + \frac{2\{u^{2n-1}-nu^{2n-2}\}}{(2n-2)!}\,G^{(r)}{}_{2n}\right]\end{aligned}\right\} \quad \ldots \quad (148)$$

Then

$$\chi_{2r+1} = 4\int_0^\infty \frac{\eta C\sin u\xi}{\Sigma'}\,(c\,\Phi - s\Psi)\,du$$

$$+\,4\int_0^\infty \frac{S\sin u\xi}{u\Sigma'}\,\{cu\Psi - (c+us)\,\Phi\}\,du$$

or, substituting for $\eta C \sin u\xi$, $S \sin u\xi$ the expansions given in (40),

$$\chi_{2r+1} = \sum_{n=1}^{\infty}\left(P^{(r)}{}_{2n} + Q^{(r)}{}_{2n}\rho^2\right)\rho^{2n}\sin 2n\theta, \quad \ldots \ldots \ldots \quad (149)$$

where

$$\left.\begin{aligned}P^{(r)}{}_{2n} &= \frac{4}{(2n)!}\int_0^\infty \left[c\,\{(n-1)\,\Phi + u\Psi\} - s\,(n\Psi + u\,\Phi)\right]\frac{u^{2n-1}}{\Sigma'}\,du\\ Q^{(r)}{}_{2n} &= \frac{2}{(2n+1)!}\int_0^\infty \frac{c\,\Phi - s\Psi}{\Sigma'}\,u^{2n+1}\,du\end{aligned}\right\} \quad \ldots \quad (150)$$

Reducing these in the same way as before, we obtain

$$\left.\begin{aligned}P^{(r)}{}_{2n} &= \sum_{p=1}^{\infty}\left\{{}^{2n}\kappa_{2p}\,F^{(r)}{}_{2p} + {}^{2n}\mu_{2p}\,G^{(r)}{}_{2p}\right\}\\ Q^{(r)}{}_{2n} &= \sum_{p=1}^{\infty}\left\{{}^{2n}\nu_{2p}\,F^{(r)}{}_{2p} + {}^{2n}\varpi_{2p}\,G^{(r)}{}_{2p}\right\}\end{aligned}\right\}, \quad \ldots \ldots \ldots \quad (151)$$

where, if $n > 1$, the coefficients conform to the definitions of (119). If $n = 1$, $Q^{(r)}{}_{2n}$ needs no special treatment, but $P^{(r)}{}_{2n}$ is not obviously convergent. We have

$$P_2^{(r)} = 2\int_0^\infty \left[c\,\{(n-1)\,\Phi + u\Psi\} - s\,(n\Psi + u\,\Phi)\right]\frac{u}{\Sigma'}\,du.$$

Now

$$\Phi = G_2 u + O(u^2)$$

$$\Psi = G_2 + O(u).$$

Hence $c\{(n-1)\ \Phi + u\Psi\} - s\,(n\Psi + u\,\Phi) = \mathrm{O}\,(u^2)$ and, since $\Sigma' = \mathrm{O}\,(u^3)$, the integral converges. Substituting the full values of Φ and Ψ, we find

$$\mathrm{P}_2^{(r)} = \int_0^\infty \Big[\tfrac{1}{2}\,(1 - 2u - e^{-2u}) \sum_{p=1}^{\infty} \frac{\mathrm{F}^{(r)}{}_{2p}\,u^{2p}}{(2p-1)\,!}$$
$$+ \sum_{p=1}^{\infty} \{p\,(2u^{2p-1} - u^{2p-2} + e^{-2u}u^{2p-2}) + 2u^{2p}\} \frac{\mathrm{G}^{(r)}{}_{2p}}{(2p-2)\,!}\Big] \frac{u}{\Sigma'}\,du$$
$$= \sum_{p=1}^{\infty} \{{}^2\kappa_{2p}\,\mathrm{F}^{(r)}{}_{2p} + {}^2\mu_{2p}\,\mathrm{G}^{(r)}{}_{2p}\},$$

where ${}^2\kappa_{2p}$ falls within the definitions (119) if $p \geq 1$ and ${}^2\mu_{2p}$ for $p > 1$. Also

$$\begin{aligned} {}^2\mu_2 &= \int_0^\infty \frac{2u^2 - u + e^{-2u}\,u}{\Sigma'}\,du - 2\mathrm{I}'_3 \\ &= 2\mathrm{I}'_2 - \mathrm{I}'_1 + \mathrm{J}'_1 - 2\mathrm{I}'_3, \qquad (152) \end{aligned}$$

where I'_1, I'_2 and J'_1 are defined as in (28). On this understanding, therefore, all the coefficients in (151) accord with the definitions in (119).

The stresses corresponding to χ_{2r+1} are

$$\left.\begin{aligned} \widehat{rr} &= -\frac{2}{b^2} \sum_{n=1}^{\infty} \{n\,(2n-1)\,\mathrm{P}^{(r)}{}_{2n} + (n-1)\,(2n+1)\,\rho^2\mathrm{Q}^{(r)}{}_{2n}\}\,\rho^{2n-2} \sin 2n\theta \\ \widehat{r\theta} &= -\frac{2}{b^2} \sum_{n=1}^{\infty} \{n\,(2n-1)\,\mathrm{P}^{(r)}{}_{2n} + n\,(2n+1)\,\rho^2\mathrm{Q}^{(r)}{}_{2n}\}\,\rho^{2n-2} \cos 2n\theta \\ \widehat{\theta\theta} &= \frac{2}{b^2} \sum_{n=1}^{\infty} \{n\,(2n-1)\,\mathrm{P}^{(r)}{}_{2n} + (n+1)\,(2n+1)\,\rho^2\mathrm{Q}^{(r)}{}_{2n}\}\,\rho^{2n-2} \sin 2n\theta \end{aligned}\right\} \qquad (153)$$

Those due to χ_{2r+2} are given by changing the upper suffix in (140) to $(r+1)$. They will cancel the $\widehat{rr}$ and $\widehat{r\theta}$ in (153) when $\rho = \lambda$ if

$$\left.\begin{aligned} \mathrm{F}_{2n}{}^{(r+1)} &= \{(2n-1)\,\mathrm{P}^{(r)}{}_{2n} + 2n\mathrm{Q}^{(r)}{}_{2n}\,\lambda^2\}\,\lambda^{4n} \\ \mathrm{G}_{2n}{}^{(r+1)} &= -\{2n\mathrm{P}^{(r)}{}_{2n} + (2n+1)\,\mathrm{Q}^{(r)}{}_{2n}\lambda^2\}\lambda^{4n-2} \end{aligned}\right\} \qquad (154)$$

The direct transformation from χ_{2r} to χ_{2r+2} is now obtainable. From (151) and (154) we have

$$\left.\begin{aligned} \mathrm{F}_{2n}{}^{(r+1)} &= \sum_{p=1}^{\infty} \{{}^{2n}s_{2p}\,\mathrm{F}^{(r)}{}_{2p} + {}^{2n}t_{2p}\,\mathrm{G}^{(r)}{}_{2p}\} \\ \mathrm{G}_{2n}{}^{(r+1)} &= \sum_{p=1}^{\infty} \{{}^{2n}u_{2p}\,\mathrm{F}^{(r)}{}_{2p} + {}^{2n}v_{2p}\,\mathrm{G}^{(r)}{}_{2p}\} \end{aligned}\right\}, \qquad (155)$$

where the coefficients are defined as in (131).

Writing

$$\left.\begin{aligned} f_{2n} &= \sum_{(r)} \mathrm{F}^{(r)}{}_{2n}\,,\; g_{2n} = \sum_{(r)} \mathrm{G}^{(r)}{}_{2n}, \\ p_{2n} &= \sum_{(r)} \mathrm{P}^{(r)}{}_{2n}\,,\; q_{2n} = \sum_{(r)} \mathrm{Q}^{(r)}{}_{2n}, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (156)^*$$

we now have from (151)

$$\left.\begin{aligned} p_{2n} &= \sum_{p=1}^{\infty} \{{}^{2n}\kappa_{2p} f_{2p} + {}^{2n}\mu_{2p}\, g_{2p}\} \\ q_{2n} &= \sum_{p=1}^{\infty} \{{}^{2n}\nu_{2p} f_{2p} + {}^{2n}\varpi_{2p}\, g_{2p}\} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (157)$$

and the complete solution is given by

$$\chi''' = \sum_{n=1}^{\infty} \left\{\frac{f_{2n}}{\rho^{2n}} + \frac{g_{2n}}{\rho^{2n-2}} + p_{2n}\rho^{2n} + q_{2n}\rho^{2n+2}\right\} \sin 2n\theta \quad . \; . \; . \; . \; . \quad (158)$$

with the stress-components

$$\left.\begin{aligned} \widehat{rr} &= -\frac{2}{b^2}\sum_{n=1}^{\infty}\left\{\frac{n\,(2n+1) f_{2n}}{\rho^{2n+2}} + \frac{(n+1)\,(2n-1)\, g_{2n}}{\rho^{2n}} + n\,(2n-1)\, p_{2n}\rho^{2n-2}\right. \\ &\qquad\qquad \left. + (n-1)\,(2n+1)\, q_{2n}\rho^{2n}\right\} \sin 2n\theta \\ \widehat{r\theta} &= \frac{2}{b^2}\sum_{n=1}^{\infty}\left\{\frac{n\,(2n+1) f_{2n}}{\rho^{2n+2}} + \frac{n\,(2n-1)\, g_{2n}}{\rho^{2n}} - n\,(2n-1)\, p_{2n}\rho^{2n-2}\right. \\ &\qquad\qquad \left. - n\,(2n+1)\, q_{2n}\rho^{2n}\right\} \cos 2n\theta \\ \widehat{\theta\theta} &= \frac{2}{b^2}\sum_{n=1}^{\infty}\left\{\frac{n\,(2n+1) f_{2n}}{\rho^{2n+2}} + \frac{(n-1)\,(2n-1)\, g_{2n}}{\rho^{2n}} + n(2n-1)\, p_{2n}\rho^{2n-2}\right. \\ &\qquad\qquad \left. + (n+1)\,(2n+1)\, q_{2n}\rho^{2n}\right\} \sin 2n\theta \end{aligned}\right\} \quad . \; . \quad (159)$$

The values of the coefficients required in the present section are given in Tables XV to XVIII.

TABLE XV.—Values of $-{}^{n}\kappa_{p}$ (even suffixes).

	$n = 2.$	$n = 4.$	$n = 6.$	$n = 8.$	$n = 10.$	$n = 12.$	$n = 14.$
$p = 2$	1·4382	0·47222	0·16261	0·052434	0·016072	0·004756	0·001373
$p = 4$	1·6675	1·3814	0·82107	0·40270	0·17452	0·069419	0·025939
$p = 6$	1·1698	1·7291	1·5801	1·0998	0·64150	0·33047	0·15525
$p = 8$	0·63620	1·4519	1·8857	1·7675	1·3354	0·86497	0·49902
$p = 10$	0·29586	0·96043	1·6803	2·0402	1·9382	1·5417	1·0716
$p = 12$	0·12383	0·54159	1·2275	1·8741	2·1865	2·0953	1·7269
$p = 14$	0·048071	0·27238	0·77626	1·4555	2·0457	2·3247	2·2417

* See footnote to equations (72).

TABLE XVI.—Values of $-{}^{n}\mu_{p}$ (even suffixes).

	$n = 2.$	$n = 4.$	$n = 6.$	$n = 8.$	$n = 10.$	$n = 12.$	$n = 14.$
$p = 2$	1·3337	0·24364	0·060588	0·015370	0·003879	0·000974	0·000244
$p = 4$	1·4619	1·2703	0·65770	0·28115	0·10738	0·038078	0·012818
$p = 6$	0·90883	1·6443	1·4779	0·96695	0·52368	0·24994	0·10894
$p = 8$	0·43036	1·3120	1·8050	1·6759	1·2219	0·75537	0·41400
$p = 10$	0·17453	0·80532	1·5710	1·9638	1·8547	1·4416	0·97032
$p = 12$	0·064275	0·41885	1·0997	1·7805	2·1144	2·0183	1·6366
$p = 14$	0·022194	0·19438	0·66094	1·3455	1·9622	2·2565	2·1697

TABLE XVII.—Values of ${}^{n}\nu_{p}$ (even suffixes).

	$n = 2.$	$n = 4.$	$n = 6.$	$n = 8.$	$n = 10.$	$n = 12.$	$n = 14.$
$p = 2$	0·72304	0·34100	0·14215	0·053877	0·019060	0·006410	0·002075
$p = 4$	1·1367	0·99508	0·64652	0·34943	0·16667	0·072637	0·029572
$p = 6$	0·99508	1·3577	1·2579	0·91667	0·56657	0·31051	0·15525
$p = 8$	0·64652	1·2579	1·5714	1·4839	1·1533	0·77626	0·46760
$p = 10$	0·34943	0·91667	1·4839	1·7620	1·6819	1·3638	0·97417
$p = 12$	0·16667	0·56657	1·1533	1·6819	1·9342	1·8598	1·5542
$p = 14$	0·072637	0·31051	0·77626	1·3638	1·8598	2·0922	2·0225

TABLE XVIII.—Values of ${}^{n}\varpi_{p}$ (even suffixes).

	$n = 2.$	$n = 4.$	$n = 6.$	$n = 8.$	$n = 10.$	$n = 12.$	$n = 14.$
$p = 2$	0·47940	0·16675	0·055707	0·017672	0·005379	0·001588	0·000458
$p = 4$	0·94444	0·82883	0·49404	0·24199	0·10477	0·041661	0·015564
$p = 6$	0·81305	1·2316	1·1287	0·78572	0·45826	0·23606	0·11089
$p = 8$	0·48938	1·1276	1·4664	1·3747	1·0386	0·67276	0·38813
$p = 10$	0·24107	0·78532	1·3746	1·6692	1·5858	1·2614	0·87675
$p = 12$	0·10463	0·45817	1·0386	1·5858	1·8501	1·7730	1·4612
$p = 14$	0·041641	0·23604	0·67275	1·2614	1·7730	2·0148	1·9428

§ 7. THE FUNDAMENTAL SOLUTIONS.

The solutions corresponding to forces transmitted from infinity have a fundamental place in the theory. Since they give zero tractions on all the boundaries, any multiple of any one of them may be added to the solution of another stress problem without altering the boundary conditions. They therefore play the part of "complementary functions." By adding them to a given solution it is possible to adjust the total tension, total shear, and total bending-moment to any prescribed values at two chosen sections of the strip, provided always that the values prescribed are consistent with the statical equilibrium of the portion between those sections.*

* *Cf.* "B," p. 101 and p. 105. See also the last section of the paper cited on p. 158 above.

We shall therefore give these solutions in full so that they may be available for use in any future applications of our method. There are three such fundamental solutions, corresponding respectively to tension, bending-moment and bending-moment with shear. They will be taken in this order.

(1) *The Tension Solution.*

The solution giving a uniform tension at infinity was fully discussed in "A," and will not be recapitulated here, but comes under discussion later (§ 8), where it is compared with an approximate solution.

(2) *The Bending-Moment Solution.*

The stress-function

$$\chi'_0 = \tfrac{1}{4}\,\mathrm{M}\eta^3 \qquad \dots \dots \dots \dots \quad (160)$$

corresponds to stress-components

$$\widehat{xx} = \frac{3\mathrm{M}\eta}{2b^2}, \quad \widehat{xy} = \widehat{yy} = 0. \qquad \dots \dots \dots \dots \quad (161)$$

If the hole is absent, the boundary conditions are satisfied; across every transverse section of the strip there is a pure bending-moment of amount

$$\frac{3\mathrm{M}}{2b^2}\int_{-1}^{1} b^2\,\eta^2\,d\eta = 3\mathrm{M}\int_0^1 \eta^2\,d\eta = \mathrm{M}.$$

To complete the solution when the hole is present, we first form the polar components of stress. These are

$$\left.\begin{aligned}\widehat{rr} &= \frac{3\mathrm{M}\rho}{8b^2}(\cos\theta - \cos 3\theta)\\ \widehat{r\theta} &= \frac{3\mathrm{M}\rho}{8b^2}(\sin 3\theta + \sin\theta)\\ \widehat{\theta\theta} &= \frac{3\mathrm{M}\rho}{8b^2}(\cos 3\theta + 3\cos\theta)\end{aligned}\right\} \qquad \dots \dots \quad (162)$$

To cancel these on the circle $\rho = \lambda$, take

$$\chi_0 = \frac{\mathrm{D}_1^{(0)}}{\rho}\cos\theta + \left(\frac{\mathrm{D}_3^{(0)}}{\rho^3} + \frac{\mathrm{E}_3^{(0)}}{\rho}\right)\cos 3\theta. \qquad \dots \dots \dots \quad (163)$$

Then it is easy to determine that

$$\mathrm{D}_1^{(0)} = \tfrac{3}{16}\,\mathrm{M}\lambda^4, \quad \mathrm{D}_3^{(0)} = \tfrac{1}{8}\,\mathrm{M}\lambda^6, \quad \mathrm{E}_3^{(0)} = -\tfrac{3}{16}\,\mathrm{M}\lambda^4. \qquad \dots \dots \quad (164)$$

It is convenient to remove the factor $\frac{1}{16}\mathrm{M}\lambda^4$ from these and subsequent coefficients by writing

$$\left.\begin{aligned}\mathrm{D}_n^{(r)} &= \tfrac{1}{16}\,\mathrm{M}\lambda^4\,d_n^{(r)}, \quad \mathrm{E}_n^{(r)} = \tfrac{1}{16}\,\mathrm{M}\lambda^4\,e_n^{(r)}\\ \mathrm{L}_n^{(r)} &= \tfrac{1}{16}\,\mathrm{M}\lambda^4\,l_n^{(r)}, \quad \mathrm{M}_n^{(r)} = \tfrac{1}{16}\,\mathrm{M}\lambda^4\,m_n^{(r)}\end{aligned}\right\} \qquad \dots \dots \quad (165)$$

so that

$$d_1^{(0)} = 3, \quad d_3^{(0)} = 2\lambda^2, \quad e_3^{(0)} = -3. \qquad \dots \dots \dots \quad (166)$$

Repeated application of (130) then gives the values of the coefficients which, on addition, lead to the complete stress-function

$$\chi = \chi'_0 + \frac{M\lambda^4}{16}\sum_{n=0}^{\infty}\left\{\frac{d_{2n+1}}{\rho^{2n+1}} + \frac{e_{2n+1}}{\rho^{2n-1}} + l_{2n+1}\,\rho^{2n+1} + m_{2n+1}\,\rho^{2n+3}\right\}\cos(2n+1)\,\theta \quad . \quad . \ (167)$$

where the coefficients are defined as before, *i.e.*, they are the sums of the corresponding coefficients in the separate stress-functions. The values of all these coefficients for values 0·1, 0·2, 0·3, 0·4, 0·5 of λ are given in Table XIX.

TABLE XIX.—Final Coefficients in the Solution for Bending-Moment.

	$\lambda = 0 \cdot 1$.	$\lambda = 0 \cdot 2$.	$\lambda = 0 \cdot 3$.	$\lambda = 0 \cdot 4$.	$\lambda = 0 \cdot 5$.
d_1	3·000	3·0006	3·0035	3·0133	3·038
d_3	$2 \cdot 00 \times 10^{-2}$	$8 \cdot 036 \times 10^{-2}$	$1 \cdot 839 \times 10^{-1}$	$3 \cdot 406 \times 10^{-1}$	$5 \cdot 742 \times 10^{-1}$
d_5	$4 \cdot 87 \times 10^{-10}$	$4 \cdot 73 \times 10^{-7}$	$2 \cdot 52 \times 10^{-5}$	$4 \cdot 05 \times 10^{-4}$	$3 \cdot 41 \times 10^{-3}$
d_7	$2 \cdot 61 \times 10^{-14}$	$4 \cdot 02 \times 10^{-10}$	$1 \cdot 06 \times 10^{-7}$	$5 \cdot 23 \times 10^{-6}$	$1 \cdot 03 \times 10^{-4}$
d_9	$1 \cdot 14 \times 10^{-18}$	$2 \cdot 77 \times 10^{-13}$	$3 \cdot 62 \times 10^{-10}$	$5 \cdot 46 \times 10^{-8}$	$2 \cdot 50 \times 10^{-6}$
e_3	$-3 \cdot 001$	$-3 \cdot 014$	$-3 \cdot 066$	$-3 \cdot 195$	$-3 \cdot 455$
e_5	$-6 \cdot 09 \times 10^{-8}$	$-1 \cdot 48 \times 10^{-5}$	$-3 \cdot 51 \times 10^{-4}$	$-3 \cdot 19 \times 10^{-3}$	$-1 \cdot 72 \times 10^{-2}$
e_7	$-3 \cdot 05 \times 10^{-12}$	$-1 \cdot 17 \times 10^{-8}$	$-1 \cdot 38 \times 10^{-6}$	$-3 \cdot 82 \times 10^{-5}$	$-4 \cdot 83 \times 10^{-4}$
e_9	$-1 \cdot 28 \times 10^{-16}$	$-7 \cdot 80 \times 10^{-12}$	$-4 \cdot 54 \times 10^{-9}$	$-3 \cdot 85 \times 10^{-7}$	$-1 \cdot 13 \times 10^{-5}$
l_3	2·988	2·926	2·855	2·811	2·836
l_5	1·228	1·195	1·151	1·111	1·095
l_7	$4 \cdot 40 \times 10^{-1}$	$4 \cdot 25 \times 10^{-1}$	$4 \cdot 05 \times 10^{-1}$	$3 \cdot 84 \times 10^{-1}$	$3 \cdot 69 \times 10^{-1}$
l_9	$1 \cdot 44 \times 10^{-1}$	$1 \cdot 38 \times 10^{-1}$	$1 \cdot 30 \times 10^{-1}$	$1 \cdot 21 \times 10^{-1}$	$1 \cdot 14 \times 10^{-1}$
l_{11}	$4 \cdot 46 \times 10^{-2}$	$4 \cdot 25 \times 10^{-2}$	$3 \cdot 95 \times 10^{-2}$	$3 \cdot 60 \times 10^{-2}$	$3 \cdot 32 \times 10^{-2}$
l_{13}	$1 \cdot 33 \times 10^{-2}$	$1 \cdot 26 \times 10^{-2}$	$1 \cdot 15 \times 10^{-2}$	$1 \cdot 03 \times 10^{-2}$	$9 \cdot 29 \times 10^{-3}$
l_{15}	$3 \cdot 85 \times 10^{-3}$	$3 \cdot 62 \times 10^{-3}$	$3 \cdot 27 \times 10^{-3}$	$2 \cdot 87 \times 10^{-3}$	$2 \cdot 58 \times 10^{-3}$
m_1	$2 \cdot 94 \times 10^{-1}$	$3 \cdot 51 \times 10^{-1}$	$4 \cdot 30 \times 10^{-1}$	$5 \cdot 20 \times 10^{-1}$	$6 \cdot 11 \times 10^{-1}$
m_3	$-1 \cdot 400$	$-1 \cdot 357$	$-1 \cdot 302$	$-1 \cdot 254$	$-1 \cdot 232$
m_5	$-8 \cdot 19 \times 10^{-1}$	$-7 \cdot 93 \times 10^{-1}$	$-7 \cdot 59 \times 10^{-1}$	$-7 \cdot 26 \times 10^{-1}$	$-7 \cdot 06 \times 10^{-1}$
m_7	$-3 \cdot 66 \times 10^{-1}$	$-3 \cdot 53 \times 10^{-1}$	$-3 \cdot 35 \times 10^{-1}$	$-3 \cdot 17 \times 10^{-1}$	$-3 \cdot 04 \times 10^{-1}$
m_9	$-1 \cdot 43 \times 10^{-1}$	$-1 \cdot 38 \times 10^{-1}$	$-1 \cdot 29 \times 10^{-1}$	$-1 \cdot 21 \times 10^{-1}$	$-1 \cdot 14 \times 10^{-1}$
m_{11}	$-5 \cdot 16 \times 10^{-2}$	$-4 \cdot 93 \times 10^{-2}$	$-4 \cdot 60 \times 10^{-2}$	$-4 \cdot 23 \times 10^{-2}$	$-3 \cdot 93 \times 10^{-2}$
m_{13}	$-1 \cdot 75 \times 10^{-2}$	$-1 \cdot 67 \times 10^{-2}$	$-1 \cdot 54 \times 10^{-2}$	$-1 \cdot 40 \times 10^{-2}$	$-1 \cdot 28 \times 10^{-2}$
m_{15}	$-5 \cdot 72 \times 10^{-3}$	$-5 \cdot 42 \times 10^{-3}$	$-4 \cdot 96 \times 10^{-3}$	$-4 \cdot 43 \times 10^{-3}$	$-3 \cdot 99 \times 10^{-3}$

Among the stress components the greatest interest attaches to $\widehat{\theta\theta}$. Its value is

$$\left.\begin{aligned}\widehat{\theta\theta} = {} & \frac{3M\lambda}{8b^2}(3\cos\theta + \cos 3\theta) \\ & + \frac{M\lambda^4}{8b^2}\sum_{n=0}^{\infty}\left\{n(2n+1)\,l_{2n+1}\rho^{2n-1} + (n+1)(2n+3)\,m_{2n+1}\rho^{2n+1}\right. \\ & \left. + \frac{(n+1)(2n+1)\,d_{2n+1}}{\rho^{2n+3}} + \frac{n(2n-1)\,e_{2n+1}}{\rho^{2n+1}}\right\}\cos(2n+1)\,\theta\end{aligned}\right\} \quad . \quad . \ (168)$$

On the edge of the hole

$$\widehat{\theta\theta} = \frac{M}{b^2} \sum_{n=0}^{\infty} S_{2n+1} \cos (2n+1)\,\theta \quad . \quad . \quad . \quad (169)$$

where

$$\left.\begin{aligned} S_1 &= \tfrac{1}{8}\lambda \{(d_1+9)+3m_1\lambda^4\} \\ S_3 &= \tfrac{1}{8}\left\{\frac{6d_3}{\lambda}+(e_3+3)\,\lambda+3l_3\lambda^5+10m_3\lambda^7\right\} \\ S_5 &= \tfrac{1}{8}\left\{\frac{15d_5}{\lambda^3}+\frac{6e_5}{\lambda}+10l_5\lambda^7+21m_5\lambda^9\right\} \\ S_7 &= \tfrac{1}{8}\left\{\frac{28d_7}{\lambda^5}+\frac{15e_7}{\lambda^3}+21l_7\lambda^9+36m_7\lambda^{11}\right\} \\ S_9 &= \tfrac{1}{8}\left\{\frac{45d_9}{\lambda^7}+\frac{28e_9}{\lambda^5}+36l_9\lambda^{11}+55m_9\lambda^{13}\right\} \end{aligned}\right\} \quad . \quad . \quad . \quad (170)$$

The values of these coefficients are shown in Table XX, and the values of $\widehat{\theta\theta}$ at the edge of the hole are given in Table XXI. It is interesting to compare these results with those obtained for the perforated strip under tension ("A," pp. 74, *et seq.*). The comparison is made in Table XXII, where the maximum stress (at $\theta = 0$) is shown by the side of (1) the tension T which would be produced at this point by the bending-moment if the hole were absent, (2) the stress which would occur at the point $\theta = 0$ on the edge of the hole if a uniform tension T were applied to the strip.

TABLE XX.

	$\lambda = 0 \cdot 1$.	$\lambda = 0 \cdot 2$.	$\lambda = 0 \cdot 3$.	$\lambda = 0 \cdot 4$.	$\lambda = 0 \cdot 5$.
S_1	$1 \cdot 50 \times 10^{-1}$	$3 \cdot 00 \times 10^{-1}$	$4 \cdot 51 \times 10^{-1}$	$6 \cdot 03 \times 10^{-1}$	$7 \cdot 60 \times 10^{-1}$
S_3	$1 \cdot 50 \times 10^{-1}$	$3 \cdot 01 \times 10^{-1}$	$4 \cdot 60 \times 10^{-1}$	$6 \cdot 37 \times 10^{-1}$	$8 \cdot 53 \times 10^{-1}$
S_5	$6 \cdot 07 \times 10^{-7}$	$7 \cdot 34 \times 10^{-5}$	$1 \cdot 15 \times 10^{-3}$	$7 \cdot 66 \times 10^{-3}$	$3 \cdot 24 \times 10^{-2}$
S_7	$4 \cdot 56 \times 10^{-9}$	$2 \cdot 19 \times 10^{-6}$	$7 \cdot 51 \times 10^{-5}$	$8 \cdot 73 \times 10^{-4}$	$5 \cdot 51 \times 10^{-3}$
S_9	$2 \cdot 57 \times 10^{-11}$	$4 \cdot 84 \times 10^{-8}$	$3 \cdot 67 \times 10^{-6}$	$7 \cdot 31 \times 10^{-5}$	$6 \cdot 89 \times 10^{-4}$

TABLE XXI.*—Values of $\frac{b^2}{M}\widehat{\theta\theta}$ at the edge of the hole.

θ	$\lambda = 0 \cdot 1$.	$\lambda = 0 \cdot 2$.	$\lambda = 0 \cdot 3$.	$\lambda = 0 \cdot 4$.	$\lambda = 0 \cdot 5$.
0°	0·30	0·61	0·91	1·25	1·65
15°	0·25	0·50	0·76	1·04	1·34
30°	0·13	0·26	0·39	0·51	0·63
45°	− 0·00	− 0·00	− 0·01	− 0·02	− 0·05
60°	− 0·08	− 0·15	− 0·23	− 0·33	− 0·46
75°	− 0·07	− 0·14	− 0·21	− 0·29	− 0·38
90°	0	0	0	0	0

* *Cf.* fig. 4, p. 213.

TABLE XXII.—Comparison of the maximum value of $\widehat{\theta\theta}$ with the stresses (1) in a non-perforated strip, (2) in a perforated tension member.

λ	Maximum value of $\widehat{\theta\theta}$.	Tension T at same point of unperforated strip under bending moment.	Maximum value of $\widehat{\theta\theta}$ in perforated strip under uniform tension T.
0·1	0·300 M/b^2	0·150 M/b^2	0·455 M/b^2
0·2	0·601	0·300	0·942
0·3	0·912	0·450	1·512
0·4	1·249	0·600	2·244
0·5	1·651	0·750	3·240

The results may be summarized by saying that the maximum value of the stress is always rather more than twice what the stress would be at the same point if the hole were absent. As, however, the stress in the unperforated strip increases linearly from the middle to the edge, the effect of the hole is not very great unless its diameter is considerable. When the hole occupies half the width of the strip, the stresses at its edge are greater than the greatest tensions which would occur in the unperforated strip at its edge. They are, however, much less than those occurring in a strip under a tension equal to that which would be produced in the unperforated strip at a distance λ from the centre by the bending-moment.

We next investigate the distribution of stress across the minimum section, $\xi = 0$. On this line $\widehat{\theta\theta}$ is given by

$$\widehat{\theta\theta} = \widehat{xx} = \frac{M}{8b^2}\left[12\rho + \lambda^4 \sum_{n=1}^{\infty} \left\{n(2n+1)(m_{2n-1} + l_{2n+1})\rho^{2n-1} + \frac{n(2n-1)}{\rho^{2n+1}}(d_{2n-1} + e_{2n+1})\right\}\right]. \quad . \quad . \quad (171)$$

This leads to the values shown in Table XXIII. From this it will be seen that, while the stress is much higher at the edge of the hole than it would be at the same point in an unperforated strip, at the edge of the strip it falls below the value occurring when the strip is unperforated.

TABLE XXIII.*—Stresses Across the Minimum Section. Values of $\frac{b^2}{M}\widehat{\theta\theta}$.

ρ	λ = 0·1.	λ = 0·2.	λ = 0·3.	λ = 0·4.	λ = 0·5.
0·1	0·30	—	—	—	—
0·2	0·30	0·60	—	—	—
0·3	0·45	0·49	0·91	—	—
0·4	0·60	0·61	0·71	1·25	—
0·5	0·75	0·75	0·79	0·97	1·65
0·6	0·90	0·90	0·92	1·00	1·27
0·7	1·05	1·05	1·06	1·10	1·24
0·8	1·20	1·20	1·21	1·23	1·30
0·9	1·35	1·35	1·35	1·35	1·38
1·0	1·50	1·50	1·49	1·46	1·42

* *Cf.* fig. 5, p. 214.

Figs. 4 and 5 illustrate the results given in Tables XXII and XXIII, and provide an easy means of comparison with the unperforated strip.

(3) *The Solution for Bending-Moment with Shear.*

The stress-function

$$\chi'_0 = \tfrac{1}{4} \mathrm{P}b\xi\eta\,(\eta^2 - 3) \quad \ldots\ldots\ldots\ldots (172)$$

corresponds to stress components

$$\widehat{xx} = \frac{3\mathrm{P}}{2b}\,\xi\eta, \quad \widehat{yy} = 0, \quad \widehat{xy} = -\frac{3\mathrm{P}}{4b}\,(\eta^2 - 1). \quad \ldots\ldots (173)$$

Integrating across the strip, we find a total shear equal to P and a bending-moment of magnitude $\mathrm{P}b\xi$ or $\mathrm{P}x$, increasing proportionately with the distance from the origin. This will be taken as the third form of fundamental solution for the unperforated strip.

In polars the function becomes

$$\chi'_0 = \tfrac{1}{32}\mathrm{P}b\,\{(2\rho^4 - 12\rho^2)\sin 2\theta + \rho^4 \sin 4\theta\} \quad \ldots\ldots (174)$$

whence

$$\left.\begin{aligned} \widehat{rr} &= \frac{3\mathrm{P}}{8b}\,(2\sin 2\theta - \rho^2 \sin 4\theta) \\ \widehat{\theta\theta} &= \frac{3\mathrm{P}}{8b}\,\{2\,(\rho^2 - 1)\sin 2\theta + \rho^2 \sin 4\theta\} \\ \widehat{r\theta} &= -\frac{3\mathrm{P}}{8b}\,\{(\rho^2 - 2)\cos 2\theta + \rho^2 \cos 4\theta\} \end{aligned}\right\} \quad \ldots\ldots (175)$$

The next stress-function χ_0 must be constructed to cancel these values of $\widehat{rr}$ and $\widehat{r\theta}$ on the circle $\rho = \lambda$. If we assume

$$\chi_0 = \left(\frac{\mathrm{F}_2^{(0)}}{\rho^2} + \mathrm{G}_2^{(0)}\right)\sin 2\theta + \left(\frac{\mathrm{F}_4^{(0)}}{\rho^4} + \frac{\mathrm{G}_4^{(0)}}{\rho^2}\right)\sin 4\theta, \quad \ldots\ldots (176)$$

the relevant stress components are

$$\widehat{rr} = -\frac{2}{b^2}\left[\left(\frac{3\mathrm{F}_2^{(0)}}{\rho^4} + \frac{2\mathrm{G}_2^{(0)}}{\rho^2}\right)\sin 2\theta + \left(\frac{10\mathrm{F}_4^{(0)}}{\rho^6} + \frac{9\mathrm{G}_4^{(0)}}{\rho^4}\right)\sin 4\theta\right]$$

$$\widehat{r\theta} = \frac{2}{b^2}\left[\left(\frac{3\mathrm{F}_2^{(0)}}{\rho^4} + \frac{\mathrm{G}_2^{(0)}}{\rho^2}\right)\cos 2\theta + \left(\frac{10\mathrm{F}_4^{(0)}}{\rho^6} + \frac{6\mathrm{G}_4^{(0)}}{\rho^4}\right)\cos 4\theta\right],$$

and these will be found to cancel the tractions on the circle $\rho = \lambda$ if

$$\left.\begin{aligned} \mathrm{F}_2^{(0)} &= -\frac{\mathrm{P}b}{8}\,\lambda^4 (3 - \lambda^2), \quad \mathrm{F}_4^{(0)} = \frac{3\mathrm{P}b\lambda^8}{32}, \\ \mathrm{G}_2^{(0)} &= \frac{3\mathrm{P}b}{16}\,\lambda^2 (4 - \lambda^2), \quad \mathrm{G}_4^{(0)} = -\frac{\mathrm{P}b\lambda^6}{8}. \end{aligned}\right\} \quad \ldots\ldots\ldots (177)$$

From these the successive sets of coefficients may be calculated, using (155). The total coefficients, with the factor Pb omitted, are given in Table XXIV.

TABLE XXIV.—Final Coefficients in the Solution for Bending-Moment with Shear.

	$\lambda = 0 \cdot 1$.	$\lambda = 0 \cdot 2$.	$\lambda = 0 \cdot 3$.	$\lambda = 0 \cdot 4$.	$\lambda = 0 \cdot 5$.
f_2	$-3\cdot84 \times 10^{-5}$	$-6\cdot58 \times 10^{-4}$	$-3\cdot75 \times 10^{-3}$	$-1\cdot40 \times 10^{-2}$	$-4\cdot27 \times 10^{-2}$
f_4	$+8\cdot84 \times 10^{-10}$	$+1\cdot83 \times 10^{-7}$	$+2\cdot80 \times 10^{-6}$	$-5\cdot8 \times 10^{-7}$	$-2\cdot42 \times 10^{-4}$
f_6	$-2\cdot21 \times 10^{-15}$	$-3\cdot67 \times 10^{-11}$	$-1\cdot06 \times 10^{-8}$	$-5\cdot88 \times 10^{-7}$	$-1\cdot33 \times 10^{-5}$
f_8	—	$-2\cdot03 \times 10^{-14}$	$-2\cdot85 \times 10^{-11}$	$-4\cdot73 \times 10^{-9}$	$-2\cdot45 \times 10^{-7}$
f_{10}	—	—	—	$-3\cdot42 \times 10^{-11}$	$-4\cdot05 \times 10^{-9}$
g_2	$7\cdot68 \times 10^{-3}$	$3\cdot30 \times 10^{-2}$	$8\cdot41 \times 10^{-2}$	$1\cdot79 \times 10^{-1}$	$3\cdot54 \times 10^{-1}$
g_4	$-1\cdot18 \times 10^{-7}$	$-6\cdot11 \times 10^{-6}$	$-4\cdot12 \times 10^{-5}$	$1\cdot03 \times 10^{-5}$	$1\cdot35 \times 10^{-3}$
g_6	$2\cdot65 \times 10^{-13}$	$1\cdot10 \times 10^{-9}$	$1\cdot41 \times 10^{-7}$	$4\cdot44 \times 10^{-6}$	$6\cdot45 \times 10^{-5}$
g_8	—	$5\cdot80 \times 10^{-13}$	$3\cdot63 \times 10^{-10}$	$3\cdot39 \times 10^{-8}$	$1\cdot13 \times 10^{-6}$
g_{10}	—	—	—	$2\cdot38 \times 10^{-10}$	$1\cdot81 \times 10^{-8}$
p_2	$-1\cdot02 \times 10^{-2}$	$-4\cdot31 \times 10^{-2}$	$-1\cdot07 \times 10^{-1}$	$-2\cdot19 \times 10^{-1}$	$-4\cdot13 \times 10^{-1}$
p_4	$-1\cdot85 \times 10^{-3}$	$-7\cdot73 \times 10^{-3}$	$-1\cdot87 \times 10^{-2}$	$-3\cdot70 \times 10^{-2}$	$-6\cdot76 \times 10^{-2}$
p_6	$-4\cdot59 \times 10^{-4}$	$-1\cdot90 \times 10^{-3}$	$-4\cdot46 \times 10^{-3}$	$-8\cdot58 \times 10^{-3}$	$-1\cdot53 \times 10^{-2}$
p_8	$-1\cdot16 \times 10^{-4}$	$-4\cdot72 \times 10^{-4}$	$-1\cdot09 \times 10^{-3}$	$-2\cdot02 \times 10^{-3}$	$-3\cdot54 \times 10^{-3}$
p_{10}	$-2\cdot92 \times 10^{-5}$	$-1\cdot17 \times 10^{-4}$	$-2\cdot62 \times 10^{-4}$	$-4\cdot72 \times 10^{-4}$	$-8\cdot17 \times 10^{-4}$
p_{12}	$-7\cdot29 \times 10^{-6}$	$-2\cdot88 \times 10^{-5}$	$-6\cdot28 \times 10^{-5}$	$-1\cdot09 \times 10^{-4}$	$-1\cdot89 \times 10^{-4}$
p_{14}	$-1\cdot82 \times 10^{-6}$	$-7\cdot08 \times 10^{-6}$	$-1\cdot49 \times 10^{-5}$	$-2\cdot49 \times 10^{-5}$	$-4\cdot41 \times 10^{-5}$
q_2	$3\cdot41 \times 10^{-3}$	$1\cdot54 \times 10^{-2}$	$3\cdot76 \times 10^{-2}$	$7\cdot57 \times 10^{-2}$	$1\cdot40 \times 10^{-1}$
q_4	$1\cdot27 \times 10^{-3}$	$5\cdot28 \times 10^{-3}$	$1\cdot27 \times 10^{-2}$	$2\cdot51 \times 10^{-2}$	$4\cdot54 \times 10^{-2}$
q_6	$4\cdot22 \times 10^{-4}$	$1\cdot74 \times 10^{-3}$	$4\cdot14 \times 10^{-3}$	$7\cdot99 \times 10^{-3}$	$1\cdot42 \times 10^{-2}$
q_8	$1\cdot34 \times 10^{-4}$	$5\cdot47 \times 10^{-4}$	$1\cdot28 \times 10^{-3}$	$2\cdot41 \times 10^{-3}$	$4\cdot24 \times 10^{-3}$
q_{10}	$4\cdot06 \times 10^{-5}$	$1\cdot65 \times 10^{-4}$	$3\cdot77 \times 10^{-4}$	$6\cdot98 \times 10^{-4}$	$1\cdot21 \times 10^{-3}$
q_{12}	$1\cdot19 \times 10^{-5}$	$4\cdot80 \times 10^{-5}$	$1\cdot08 \times 10^{-4}$	$1\cdot96 \times 10^{-4}$	$3\cdot39 \times 10^{-4}$
q_{14}	$3\cdot44 \times 10^{-6}$	$1\cdot37 \times 10^{-5}$	$3\cdot02 \times 10^{-5}$	$5\cdot34 \times 10^{-5}$	$9\cdot29 \times 10^{-5}$

In terms of these coefficients the final stress function is

$$\chi = \chi'_0 + \mathrm{P}b \sum_{n=1}^{\infty} \left\{\frac{f_{2n}}{\rho^{2n}} + \frac{g_{2n}}{\rho^{2n-2}} + p_{2n}\rho^{2n} + q_{2n}\rho^{2n+2}\right\} \sin 2n\theta, \quad . \quad . \quad . \quad (178)$$

and the corresponding stresses can be written down immediately from (159), together with (175). The value of $\widehat{\theta\theta}$ is

$$\widehat{\theta\theta} = \frac{3\mathrm{P}}{8b}\{2(\rho^2 - 1)\sin 2\theta + \rho^2 \sin 4\theta\}$$
$$+ \frac{2\mathrm{P}}{b}\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)f_{2n}}{\rho^{2n+2}} + \frac{(n-1)(2n-1)g_{2n}}{\rho^{2n}}\right.$$
$$\left. + n(2n-1)p_{2n}\rho^{2n-2} + (n+1)(2n+1)q_{2n}\rho^{2n}\right\}\sin 2n\theta. \quad . \quad . \quad (179)$$

At the edge of the hole, $\rho = \lambda$, this becomes

$$\widehat{\theta\theta} = \frac{\mathrm{P}}{b}\sum_{n=1}^{\infty} \mathrm{A}_{2n} \sin 2n\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

where

$$\left.\begin{aligned} A_2 &= \tfrac{3}{4}(\lambda^2 - 1) + \frac{6f_2}{\lambda^4} + 2p_2 + 12q_2\lambda^2 \\ A_4 &= \tfrac{3}{8}\lambda^2 + \frac{20f_4}{\lambda^6} + \frac{6g_4}{\lambda^4} + 12p_4\lambda^2 + 30q_4\lambda^4 \\ A_6 &= \frac{42f_6}{\lambda^8} + \frac{20g_6}{\lambda^6} + 30p_6\lambda^4 + 56q_6\lambda^6 \\ A_8 &= \frac{72f_8}{\lambda^{10}} + \frac{42g_8}{\lambda^8} + 56p_8\lambda^6 + 90q_8\lambda^8 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

The values of these coefficients for values of λ from 0·1 to 0·5 are shown in Table XXV and the values of $\widehat{\theta\theta}$ are given in Table XXVI.

TABLE XXV.

	$\lambda = 0{\cdot}1$.	$\lambda = 0{\cdot}2$.	$\lambda = 0{\cdot}3$.	$\lambda = 0{\cdot}4$.	$\lambda = 0{\cdot}5$.
A_2	− 3·07	− 3·27	− 3·63	− 4·20	− 5·07
A_4	0·014	0·05	0·06	0·01	− 0·20
A_6	—	—	—	− 0·02	− 0·08
A_8	—	—	—	—	− 0·01

TABLE XXVI.*—Values of $-\dfrac{b}{P}\,\widehat{\theta\theta}$ at the edge of the hole.

θ	$\lambda = 0{\cdot}0$.	$\lambda = 0{\cdot}1$.	$\lambda = 0{\cdot}2$.	$\lambda = 0{\cdot}3$.	$\lambda = 0{\cdot}4$.	$\lambda = 0{\cdot}5$.
0°	0	0	0	0	0	0
15°	1·50	1·52	1·59	1·76	2·11	2·80
30°	2·60	2·65	2·79	3·09	3·63	4·56
45°	3·00	3·07	3·27	3·63	4·18	4·99
60°	2·60	2·66	2·88	3·20	3·65	4·23
75°	1·50	1·55	1·68	1·87	2·13	2·43
90°	0	0	0	0	0	0

* *Cf.* fig. 6, p. 216.

These results, which are illustrated in fig. 6, are of considerable interest. The maximum stress occurs in all cases at about the same point, $\theta = 45°$, and is remarkably high, rising from six times the average shear at infinity to ten times as the size of the hole is increased up to half the width of the strip.†

† See discussion in § 9 below.

The other matter of principal interest is the mode of distribution of the shear across the minimum section, $\theta = 0$. The formula for $\widehat{r\theta}$ is

$$\widehat{r\theta} = -\frac{3P}{8b}\left\{(\rho^2 - 2)\cos 2\theta + \rho^2 \cos 4\theta\right\} + \frac{2P}{b}\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)f_{2n}}{\rho^{2n+2}} + \frac{n(2n-1)g_{2n}}{\rho^{2n}} - n(2n-1)p_{2n}\rho^{2n-2} - n(2n+1)q_{2n}\rho^{2n}\right\}\cos 2n\theta. \quad \ldots \quad (182)$$

When θ is put $= 0$ this becomes

$$\widehat{r\theta} = \frac{P}{b}\Big[\tfrac{3}{4}(1-\rho^2) - 2p_2 - 6(2p_4 + q_2)\rho^2 - 10(3p_6 + 2q_4)\rho^4 - 14(4p_8 + 3q_6)\rho^6 - 18(5p_{10} + 4q_8)\rho^8 - \ldots + \frac{2g_2}{\rho^2} + \frac{6(f_2 + 2g_4)}{\rho^4} + \frac{10(2f_4 + 3g_6)}{\rho^6} + \frac{14(3f_6 + 4g_8)}{\rho^8} + \ldots\Big] \quad \ldots \quad (183)$$

From this have been calculated the values of $\widehat{r\theta}$ shown in Table XXVII and illustrated in fig. 7.

TABLE XXVII.*—Values of the shear, $\frac{b}{P}\widehat{r\theta}$, on the minimum section.

ρ	$\lambda = 0 \cdot 0$.	$\lambda = 0 \cdot 1$.	$\lambda = 0 \cdot 2$.	$\lambda = 0 \cdot 3$.	$\lambda = 0 \cdot 4$.	$\lambda = 0 \cdot 5$.
0·0	0·00†	—	—	—	—	—
0·1	0·74	0·00	—	—	—	—
0·2	0·72	0·98	0·00	—	—	—
0·3	0·68	0·84	1·00	0·00	—	—
0·4	0·63	0·74	0·97	0·98	0·00	—
0·5	0·56	0·64	0·84	1·07	1·06	0·00
0·6	0·48	0·54	0·71	0·96	1·21	1·24
0·7	0·38	0·43	0·57	0·80	1·11	1·43
0·8	0·27	0·31	0·41	0·60	0·88	1·24
0·9	0·14	0·17	0·23	0·34	0·52	0·79
1·0	0·00	0·00	0·00	0·00	0·00	0·00

* *Cf.* fig. 7, p. 217.

† If there is no hole this value becomes 0·75 ; $\lambda = 0 \cdot 0$ is to be taken as meaning a very small hole, or a hole in a very wide strip.

When there is no hole the distribution of shear across the section is parabolic, rising steadily from zero at the edge to a maximum at the middle. This maximum is one and a half times the average value across the section. The presence of the hole reduces the shear to zero at the edge of the hole, and the maximum now occurs at a distance from the hole which, for moderate sizes of hole, is about $\frac{1}{10}$ b or $\frac{1}{20}$ of the whole width of the strip. For larger holes the maximum is displaced further into the body of the material. The size of the maximum increases less than in inverse ratio to the breadth of material.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

It rises from 1·5 to 2·86 times the average shear across the whole width of the strip as the size of the hole is increased from zero to half the width of the strip. It is always very much smaller than the greatest value of $\widehat{\theta\theta}$ at the edge of the hole.

§ 8. Comparison with an Approximate Solution.

Before making a final summary of our results, we shall compare them with certain approximate solutions, which may be obtained by much simpler analysis. This will serve both to throw into relief the principal features of the stress distributions under discussion, and to show how far our more elaborate analysis has justified itself by revealing new features not discoverable without its aid.*

An approximate solution of the tension problem in a perforated strip was given by G. Kirsch,† and a similar solution for bending-moment by Z. Tuzi.‡ A solution of the same type for bending-moment with shear was communicated to us by Professor Filon. All these methods are equivalent to taking for the stress-function, in the notation of the present paper,

$$\chi = \chi'_0 + \chi_0 \qquad (184)$$

and ignoring the residual stresses on the straight boundaries. They may therefore be expected to be fairly accurate for small values of λ, but to show divergences from the true results as λ increases. We proceed to examine the magnitude and nature of these divergences.

The Approximate Tension Solution.—We have§

$$\chi'_0 = \tfrac{1}{4}b^2\,\mathrm{P}\rho^2\,(1 + \cos 2\theta) \qquad (185)$$

$$\begin{aligned}\chi_0 &= -\,\mathrm{D}_0^{(0)}\log\rho + \left(\frac{\mathrm{D}_2^{(0)}}{\rho^2} + \mathrm{E}_2^{(0)}\right)\cos 2\theta \\ &= -\,\tfrac{1}{4}\,b^2\mathrm{T}\left[2\lambda^2\log\rho + \left(2\lambda^2 - \frac{\lambda^4}{\rho^2}\cos 2\theta\right)\right]\end{aligned} \qquad (186)$$

the corresponding stresses being

$$\left.\begin{aligned}\widehat{rr} &= \frac{\mathrm{T}}{2}\left[\left(1 - \frac{\lambda^2}{\rho^2}\right) - \left(1 - 4\,\frac{\lambda^2}{\rho^2} + 3\,\frac{\lambda^4}{\rho^4}\right)\cos 2\theta\right] \\ \widehat{\theta\theta} &= \frac{\mathrm{T}}{2}\left[\left(1 + \frac{\lambda^2}{\rho^2}\right) + \left(1 + 3\,\frac{\lambda^4}{\rho^4}\right)\cos 2\theta\right] \\ \widehat{r\theta} &= \frac{\mathrm{T}}{2}\left[1 + 2\frac{\lambda^2}{\rho^2} - 3\,\frac{\lambda^4}{\rho^4}\right]\sin 2\theta,\end{aligned}\right\} \qquad (187)$$

as stated by Kirsch.

* We are indebted to Professor L. N. G. Filon for suggesting this very illuminating comparison.

† 'Z. Ver. deuts. Ing.,' vol. 32, p. 797 (1898).

‡ 'Phil. Mag.,' vol. 9, p. 210 (1930) or *cf.* Coker and Filon, "Photo-elasticity," § 5·22.

§ "A," p. 69, equations (54) to (56).

Putting $\rho = \lambda$ we get, for the stress round the hole,

$$\widehat{\theta\theta}_{\rho=\lambda} = \mathrm{T}\,(1 + 2\cos 2\theta). \qquad (188)$$

This is independent of λ, and gives the following numerical values—

θ	0°	15°	30°	45°	60°	75°	90°
$\widehat{\theta\theta}/\mathrm{T}$	3·00	2·73	2·00	1·00	0·00	— 0·73	— 1·00.

A comparison with the more exact values* shows that, while when $\lambda = 0\cdot1$ the greatest discrepancy observed for any value of θ is only 3 per cent., this discrepancy rises with

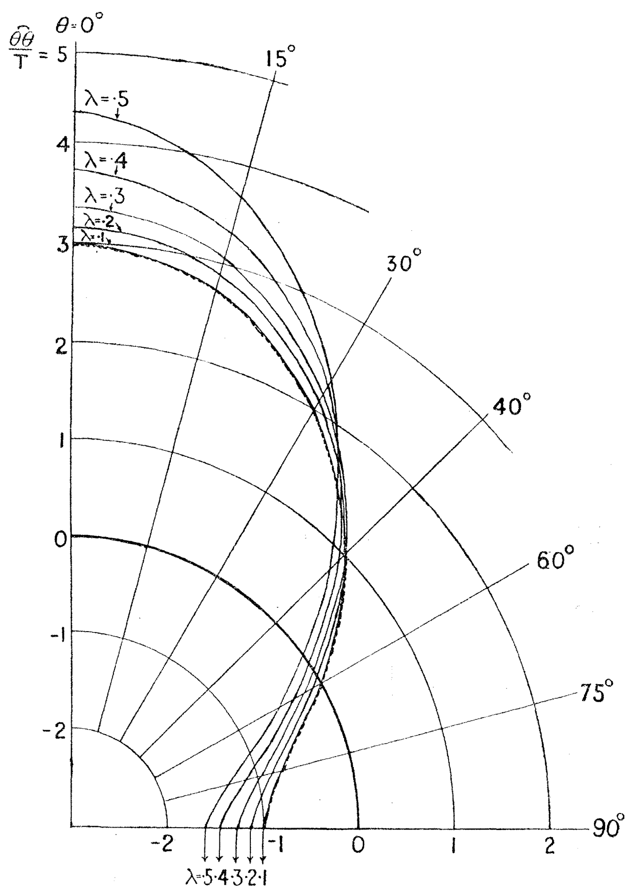

FIG. 2.—*Tension Solution.*—Values of $\widehat{\theta\theta}/\mathrm{T}$ round the hole. The dotted curve shows the values given by the approximate solution. Positive values are measured radially outwards, negative values inwards, from the circle of reference.

λ until, when $\lambda = 0\cdot5$, the true values of $\widehat{\theta\theta}$ are about one and a half times the approximate values. These results are illustrated in fig. 2, which is an elaboration of the figure in the previous paper.

* "A," p. 74, Table XII and p. 83, fig. 2.

A further interesting comparison is given by the stresses across the minimum section. The approximate solution gives for the tension

$$\widehat{xx} = (\widehat{\theta\theta})_{\theta=0} = \mathrm{T}\left[1 + \tfrac{1}{2}\frac{\lambda^2}{\rho^2} + \tfrac{3}{2}\frac{\lambda^4}{\rho^4}\right]. \qquad (189)$$

This leads to Table XXVIII for comparison with the exact values previously published.*

TABLE XXVIII.—Approximate values of $\frac{\widehat{xx}}{\mathrm{T}}$ across the minimum section.

ρ	λ = 0·1.	λ = 0·2.	λ = 0·3.	λ = 0·4.	λ = 0·5.
0·1	3·00	—	—	—	—
0·2	1·22	3·00	—	—	—
0·3	1·07	1·52	3·00	—	—
0·4	1·04	1·22	1·76	3·00	—
0·5	1·02	1·12	1·37	1·93	3·00
0·6	1·02	1·07	1·22	1·52	2·07
0·7	1·01	1·05	1·14	1·32	1·65
0·8	1·01	1·04	1·10	1·22	1·42
0·9	1·01	1·03	1·07	1·16	1·30
1·0	1·01	1·02	1·06	1·12	1·22

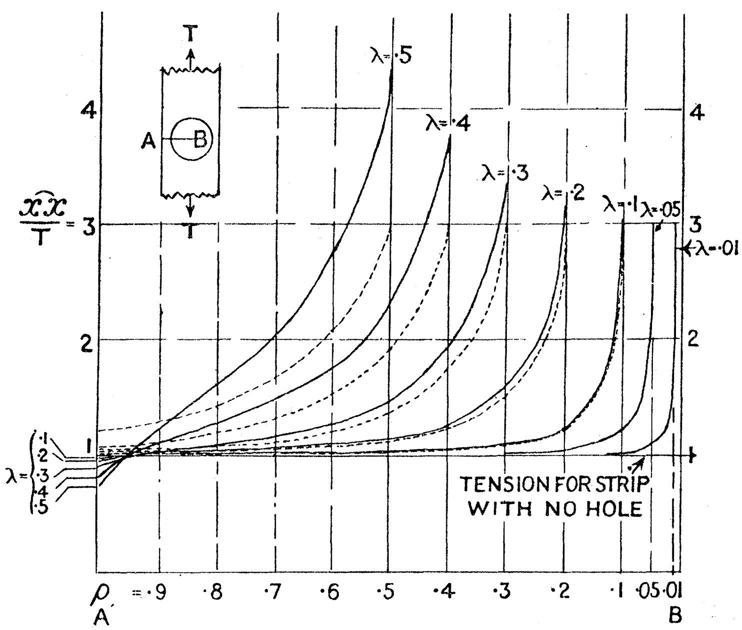

FIG. 3.—*Tension Solution.*—Values of $\widehat{xx}/\mathrm{T}$ at points on the mid-section. The dotted curves show the values given by the approximate solution.

The comparison† is shown graphically in fig. 3. For small values of λ the agreement is good, but when λ = 0·5 the exact values are more than one and a third times the

* "A," p. 76, Table XIV. (The value for λ = 0·3, ρ = 0·9 in this Table should be corrected to 1·04.)

† For a comparison with experimental results see "A" pp. 82–84, or COKER and FILON, "Photo-elasticity," p. 487.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

approximate values in the neighbourhood of the hole; but the most interesting feature of the comparison is that, whereas on the approximate theory $\widehat{xx}$ is an essentially increasing function of λ, the true value of $\widehat{xx}$ actually decreases with λ, when ρ has values near to 1. Thus the approximate solution not only gives stresses which are much too small, but it is also qualitatively in error, since it fails entirely to reveal a most interesting feature of the field of stress. A similar failure will be found in the other approximate solutions.

The Approximate Bending-Moment Solution.—In this case we have

$$\left.\begin{aligned} \chi'_0 &= \frac{M}{16}\rho^3\,(3\cos\theta + \cos 3\theta) \\ \chi_0 &= \frac{M\lambda^3}{16}\left\{3\,\frac{\lambda}{\rho}\cos\theta + \left(2\,\frac{\lambda^3}{\rho^3} - 3\,\frac{\lambda}{\rho}\right)\cos 3\theta\right\} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; (190)$$

and the stresses are

$$\left.\begin{aligned} \widehat{rr} &= \frac{3}{8}\frac{M\rho}{b^2}\left\{\left(1 - \frac{\lambda^4}{\rho^4}\right)\cos\theta - \left(1 + 4\,\frac{\lambda^6}{\rho^6} - 5\,\frac{\lambda^4}{\rho^4}\right)\cos 3\theta\right\} \\ \widehat{r\theta} &= \frac{3}{8}\frac{M\rho}{b^2}\left\{\left(1 - \frac{\lambda^4}{\rho^4}\right)\sin\theta + \left(1 - 4\,\frac{\lambda^6}{\rho^6} + 3\,\frac{\lambda^4}{\rho^4}\right)\sin 3\theta\right\} \\ \widehat{\theta\theta} &= \frac{3}{8}\frac{M\rho}{b^2}\left\{\left(3 + \frac{\lambda^4}{\rho^4}\right)\cos\theta + \left(1 + 4\,\frac{\lambda^6}{\rho^6} - \frac{\lambda^4}{\rho^4}\right)\cos 3\theta\right\} \end{aligned}\right\} \; . \; . \; . \; (191)$$

On the rim of the hole, $\rho = \lambda$, we find

$$\frac{b^2}{M}\,\widehat{\theta\theta} = \frac{3}{2}\,\lambda\,(\cos\theta + \cos 3\theta), \quad . \; . \; . \; . \; . \; . \; . \; . \; (192)$$

agreeing with TUZI's result (*loc. cit.*), and giving Table XXIX, for comparison with Table XXI.

TABLE XXIX.—Approximate Values of $\frac{b^2}{M}\widehat{\theta\theta}$ at the rim of the hole.

θ	$\lambda = 0{\cdot}1$.	$\lambda = 0{\cdot}2$.	$\lambda = 0{\cdot}3$.	$\lambda = 0{\cdot}4$.	$\lambda = 0{\cdot}5$.
0°	0·300	0·600	0·900	1·200	1·500
15°	0·251	0·502	0·753	1·004	1·255
30°	0·130	0·260	0·390	0·520	0·650
45°	0	0	0	0	0
60°	− 0·075	− 0·150	− 0·225	− 0·300	− 0·375
75°	− 0·067	− 0·134	− 0·202	− 0·269	− 0·336
90°	0	0	0	0	0

These values agree remarkably well with the exact values up to $\lambda = 0 \cdot 3$ and even when $\lambda = 0 \cdot 5$ the approximate solution gives 90 per cent. of the accurate value (*cf.* fig. 4.) The

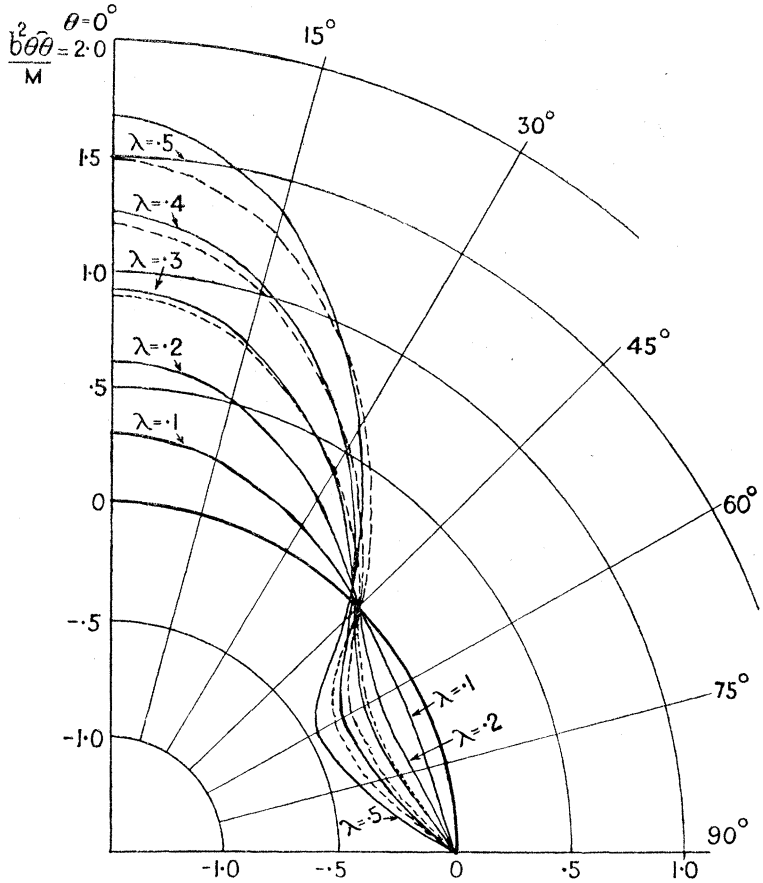

FIG. 4.—*Bending-Moment Solution.*—Values of $b^2\,\widehat{\theta\theta}/\mathrm{M}$ round the hole. The dotted curves show the values given by the approximate solution.

same degree of approximation is found in the values of $\widehat{xx}$ across the minimum section. This is given by

$$\widehat{xx} = (\widehat{\theta\theta})_{\theta=0} = \frac{3\mathrm{M}}{2b^2}\,\rho\left(1 + \frac{\lambda^6}{\rho^6}\right) \quad \ldots \ldots \ldots \quad (193)$$

from which we have Table XXX, for comparison with Table XXIII.

TABLE XXX.—Approximate Values of $\frac{b^2}{\mathrm{M}}\,\widehat{xx}$ across the minimum section, $x = 0$.

ρ	λ = 0·1.	λ = 0·2.	λ = 0·3.	λ = 0·4.	λ = 0·5.
0·1	0·300	—	—	—	—
0·2	0·305	0·600	—	—	—
0·3	0·451	0·490	0·900	—	—
0·4	0·600	0·609	0·707	1·200	—
0·5	0·750	0·753	0·785	0·947	1·500
0·6	0·900	0·901	0·914	0·979	1·201
0·7	1·050	1·051	1·057	1·087	1·189
0·8	1·200	1·200	1·203	1·219	1·272
0·9	1·350	1·350	1·352	1·360	1·390
1·0	1·500	1·500	1·501	1·506	1·523

A comparison* of the two sets of values is made in fig. 5. The comparative accuracy of the approximate solution in this case is clearly owing to the fact that the hole is

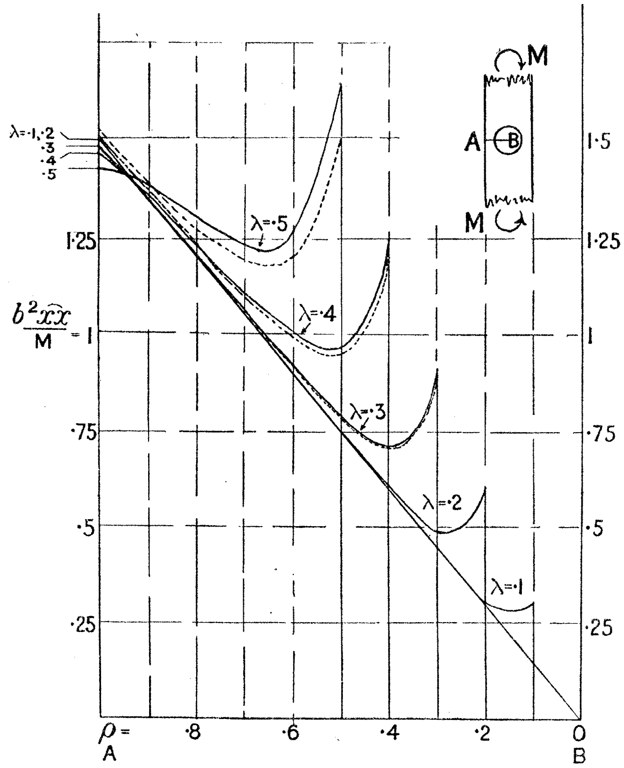

FIG. 5.—*Bending-Moment Solution.*—Values of $b^2 \widehat{xx}/M$ on the mid-section. The dotted curves show the values given by the approximate solution.

placed across the weaker parts of the field of stress. In consequence the disturbing effect at the edges of the strip is small, and this is reflected in the analysis as a rapid convergence of the χ-series. Only for larger values of λ would the exact method show divergences from the approximate values comparable with those seen in the tension problem. Fig. 5, however, shows an interesting feature of the true values, which is likely to be more pronounced for higher values of λ, and which is entirely missing from the approximate solution. The curves for the approximate values have an asymptote in common, and for values of ρ approaching 1 the true curves cross this asymptote. This results in a decrease of the stress with increase of λ, exactly as in the tension solution.

The Solution for Bending-Moment with Shear.—The exact solution for this case shows slower convergence in the χ-series than in either of the other problems, and it is therefore to be expected that the approximate solution will be even less adequate than in the

* For a comparison with experimental results by stress-optical methods see Z. TUZI, 'Sci. Pap. Inst. Phys. & Chem. Res.,' Tokyo, No. 156 (1928) and 'Phil. Mag.' vol. 9, p. 210 (1930); also COKER and FILON, "Photo-elasticity," § 5·22.

case of tension. The solution corresponding to those of KIRSCH and TUZI is given by the sum of the two stress-functions

$$\left.\begin{aligned} \chi'_0 &= \frac{Pb}{32}\{2(\rho^4 - 6\rho^2)\sin 2\theta + \rho^4 \sin 4\theta\} \\ \chi_0 &= \frac{Pb}{32}\left\{\left[6\lambda^2(4-\lambda^2) - 4\lambda^2(3-\lambda^2)\frac{\lambda^2}{\rho^2}\right]\sin 2\theta + \left[3\frac{\lambda^8}{\rho^4} - 4\frac{\lambda^6}{\rho^2}\right]\sin 4\theta\right\} \end{aligned}\right\}, \quad (194)$$

the corresponding stresses being

$$\left.\begin{aligned} \widehat{rr} &= \frac{3P}{8b}\left\{\left[2 + 2(3-\lambda^2)\frac{\lambda^4}{\rho^4} - 2(4-\lambda^2)\frac{\lambda^2}{\rho^2}\right]\sin 2\theta - \rho^2\left[1 + 5\frac{\lambda^8}{\rho^8} - 6\frac{\lambda^6}{\rho^6}\right]\sin 4\theta\right\} \\ \widehat{r\theta} &= \frac{3P}{8b}\left\{\left[2-\rho^2-2(3-\lambda^2)\frac{\lambda^4}{\rho^4} + (4-\lambda^2)\frac{\lambda^2}{\rho^2}\right]\cos 2\theta - \rho^2\left[1-5\frac{\lambda^8}{\rho^8} + 4\frac{\lambda^6}{\rho^6}\right]\cos 4\theta\right\} \\ \widehat{\theta\theta} &= \frac{3P}{8b}\left\{\left[2\rho^2-2-2(3-\lambda^2)\frac{\lambda^4}{\rho^4}\right]\sin 2\theta + \rho^2\left[1 + 5\frac{\lambda^8}{\rho^8} - 2\frac{\lambda^6}{\rho^6}\right]\sin 4\theta\right\} \end{aligned}\right\}. \quad (195)$$

On the boundary of the hole $\rho = \lambda$, we have

$$\widehat{\theta\theta} = \frac{3P}{2b}\{(\lambda^2 - 2)\sin 2\theta + \lambda^2 \sin 4\theta\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (196)$$

and this leads to Table XXXI, which should be compared with Table XXVI.

TABLE XXXI.—Approximate Values of $-\frac{b}{P}\widehat{\theta\theta}$ at the edge of the hole.

θ	$\lambda = 0.$	$\lambda = 0 \cdot 1.$	$\lambda = 0 \cdot 2.$	$\lambda = 0 \cdot 3.$	$\lambda = 0 \cdot 4.$	$\lambda = 0 \cdot 5.$
0°	0	0	0	0	0	0
15°	1·50	1·48	1·42	1·32	1·17	0·99
30°	2·60	2·57	2·49	2·36	2·18	1·95
45°	3·00	2·98	2·94	2·86	2·76	2·62
60°	2·60	2·60	2·60	2·60	2·60	2·60
75°	1·50	1·51	1·52	1·55	1·59	1·64
90°	0	0	0	0	0	0

The comparison is made graphically in fig. 6, from which it is clear that the agreement is satisfactory only for quite small values of λ. Not only are there large discrepancies for the larger values of λ, but the approximate solution gives an altogether wrong idea of what occurs. For, whereas the approximate solution shows the stresses *decreasing* with λ for $0 < \theta < 60°$, and *increasing* for $60° < \theta < 90°$, the exact solution shows them *increasing* with λ over the whole range $0 < \theta < 90°$. Moreover, the λ gradient is reversed so decisively that the maximum stress for $\lambda = 0 \cdot 5$ is nearly twice that for $\lambda = 0 \cdot 0$, while the approximate solution actually shows it as 13 per cent. *less*.

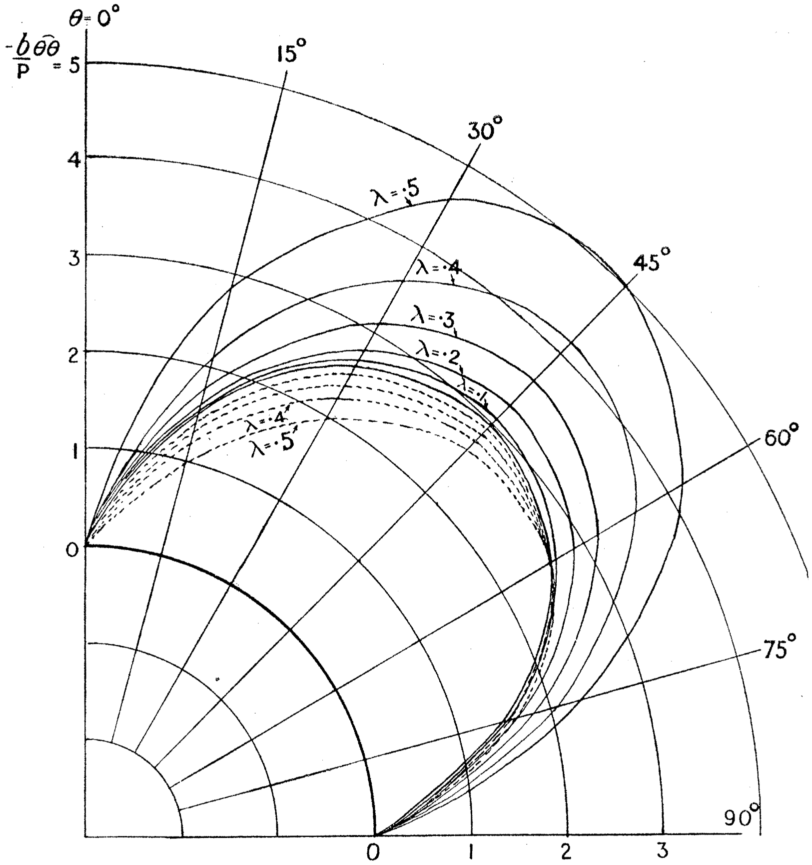

FIG. 6.—*Solution for Bending-Moment with Shear.*—Values of $-b\,\widehat{\theta\theta}/P$ round the hole. The dotted curves show the values given by the approximate solution.

We next examine the shear across the minimum section. This is given by

$$\widehat{xy} = (\widehat{r\theta})_{\theta=0} = \frac{3P}{8b}\left\{2(1-\rho^2) + (4-\lambda^2)\frac{\lambda^2}{\rho^2} - 2(3+\lambda^2)\frac{\lambda^4}{\rho^4} + 5\frac{\lambda^8}{\rho^6}\right\} \quad . \quad . \quad . \quad (197)$$

and results in Table XXXII, which is to be compared with Table XXVII.

TABLE XXXII.—Approximate Values of $\frac{b}{P}\widehat{xy}$ across the minimum section.

ρ	λ = 0·0.*	λ = 0·1.	λ = 0·2.	λ = 0·3.	λ = 0·4.	λ = 0·5.
0·0	0·00	—	—	—	—	—
0·1	0·74	0·00	—	—	—	—
0·2	0·72	0·95	0·00	—	—	—
0·3	0·68	0·82	0·90	0·00	—	—
0·4	0·63	0·71	0·86	0·75	0·00	—
0·5	0·56	0·62	0·74	0·80	0·60	0·00
0·6	0·48	0·52	0·62	0·70	0·68	0·44
0·7	0·38	0·41	0·49	0·57	0·61	0·53
0·8	0·27	0·29	0·35	0·43	0·49	0·48
0·9	0·14	0·16	0·21	0·28	0·34	0·36
1·0	0·00	0·02	0·06	0·11	0·17	0·21

* *i.e.* λ very small. If there is no hole the entry for ρ = 0·0 in this column becomes 0·75.

The last row in this table shows the shear on the boundary at the mid-section, which should be zero, and gives an idea of the magnitude of the residual stresses neglected. In the face of this, close agreement with the exact values cannot be expected. In fact, as fig. 7 shows, the errors increase rapidly after $\lambda = 0\cdot2$, until for $\lambda = 0\cdot5$ the actual

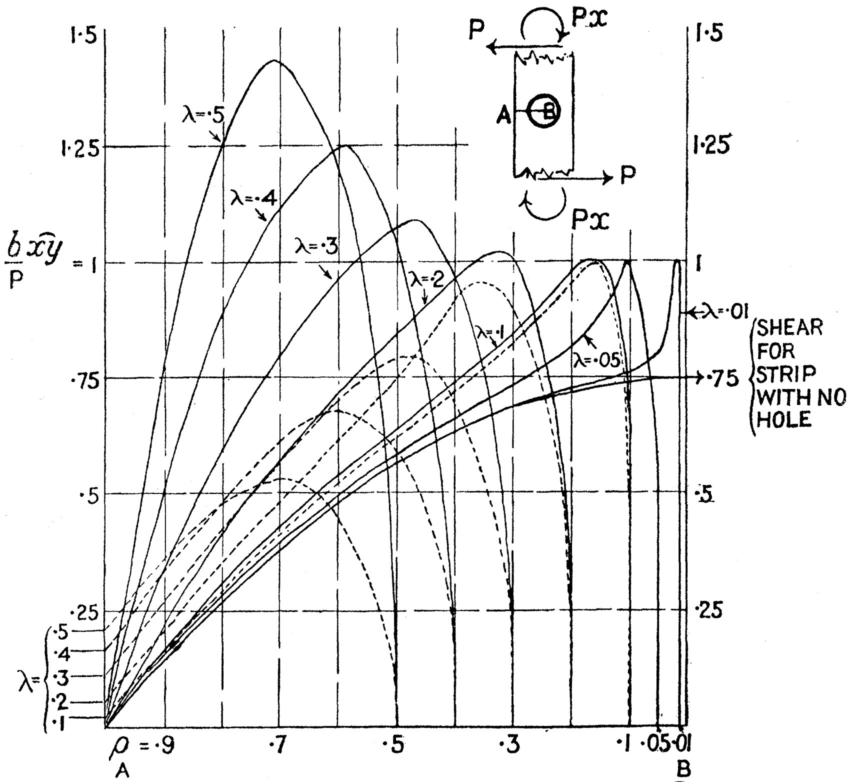

FIG. 7.—*Solution for Bending-Moment with Shear.*—Values of the shear $b\,\widehat{xy}/\mathrm{P}$ on the mid-section. The dotted curves show the values given by the approximate solution.

maximum shear is almost three times that given by the approximate solution. Part of this discrepancy is accounted for by the fact that the approximate solution only allows for a fraction of the shear across the diminished section. The whole shear across the section $x = 0$ is given by

$$2\int_{\lambda}^{1} \widehat{xy} b\, d\rho$$

which is easily found to be

$$\tfrac{1}{4}\mathrm{P}\,(1 - \lambda^2)^2\,(4 - 4\lambda^2 - 3\lambda^4).$$

When λ is small this is very nearly equal to P, but when $\lambda = 0\cdot5$ it reduces to about $0\cdot4$ P. We might, then, attempt to adjust the solution in the neighbourhood of the hole by multiplying by a factor ($2\cdot5$ in the case of $\lambda = 0\cdot5$) which will restore the total shear across the mid-section to its true value.* This would bring up the maximum shear on the section to within a little of its true value, but, besides upsetting the conditions at a distance, it would also fail entirely to account for the distribution of $\widehat{\theta\theta}$

* *Cf.* COKER, CHAKKO and SATAKE, "Photo-Elastic and Strain Measurements of the Effects of Circular Holes, etc.," 'Trans. Inst. Eng. Shipb. Scot.,' vol. 63, p. 34 (1920).

round the hole. It would seem then that, except for quite small values of λ, the approximate method can give no adequate description of the stress field, and our more elaborate analysis is fully justified.

As this appears to be the first published account, even by the approximate method, of the case of bending-moment with shear, it seems worth while to give some attention to a phenomenon occurring near a small hole. It is clear from fig. 7 that, as λ tends to 0, the maximum shear on the mid-section tends to 1, and also moves up to the edge of the hole. The distribution of stress is clearly shown by the two additional curves for $\lambda = 0 \cdot 01$ and $\lambda = 0 \cdot 05$. As we pass across the strip the distribution is almost exactly the parabolic one that occurs when there is no hole, so long as the hole is not approached closely. Then it rises very rapidly from about $0 \cdot 75$ to 1 and falls, still more rapidly, to zero. The presence of a pin-hole thus increases the maximum shear by a third, but the resulting maximum is still itself only a third of the maximum value of $\widehat{\theta\theta}$ at the edge of the hole. These phenomena are analogous to the trebling of the tension in the neighbourhood of a small hole when the strip is stretched.

To sum up the results of this section, we may say that the approximate method gives reasonably good results in the bending-moment problem for values of λ up to $0 \cdot 5$, but in the other two cases it gives values for the stresses, which are much too low, except when λ is quite small. In all three cases it fails entirely to reveal some of the most interesting features.

§ 9. Further Discussion of the Results.

The principal results of our discussion of the three fundamental stress distributions may be summarized very briefly as follows :—

(1) When a tension is applied to the strip there appears at the edge of the hole a tension, which is always more than three times the tension at infinity, and rises to four and one third times this tension when the hole occupies half the width of the strip. On and near the edge of the strip at the mid-section the tension *decreases* as the size of the hole *increases* (" A," p. 76, *et seq*.).

(2) When a bending-moment is applied, the tension at the rim of the hole is always about twice as great as it would be at the same point if the hole were absent. It is not, however, until the diameter of the hole becomes about half that of the strip that the tension near the hole exceeds that on the edge of the strip. For smaller holes the distribution of tension across the strip is very little affected, except in the immediate neighbourhood of the hole (*cf*. fig. 4). Near the edge of the strip the tension again decreases as the size of the hole increases.

The point on the rim of the hole at which the tension vanishes remains almost exactly at $\theta = 45°$ for all sizes of hole considered.

(3) Various combinations of shear with bending-moment are possible. In the one considered above there is no bending-moment across the mid-section, while the bending-

moment is small in the whole neighbourhood of the strip. The stresses found may thus be taken as the effect of the shear. At a distance from the hole the maximum shear on any cross-section is one and a half times the average shear. This value is nearly doubled on the mid-section when the hole occupies half the width of the strip. But at the edge of the hole occurs a tension three and a half times as great as the greatest shear. The allowable tension in a metal is always greater than the allowable shear but in a ratio usually less than 3 : 2. It follows that the tension at the edge of the hole is by far the most dangerous stress and failure, if it occurs, may be expected here.

Other stress systems may be obtained by superposing solutions of the above types. For example, if a solution for a bending-moment $-4\text{P}b$ is added to that for a shear P with its attendant bending-moment, this moment is cancelled at $x = 4b$, *i.e.*, at a distance from the hole equal to twice the width of the strip. At this distance the local effects of the hole may be expected to be insensible* and the action across the section will consist of a shear P together with a self-equilibrating set of very small tensions and compressions. We may thus regard the strip as cut at this point and loaded there with a transverse force P balanced at some section on the negative side of the origin by an appropriate force and couple. These conditions are illustrated in fig. 8. The corres-

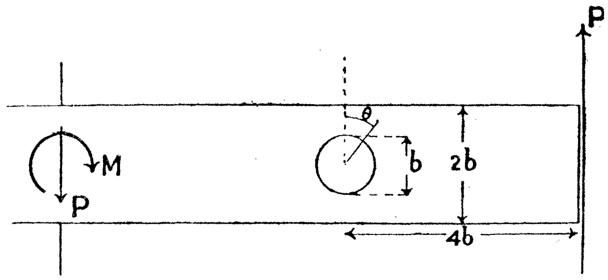

FIG. 8.

ponding stress function can be immediately written down from (167) and (178) while the more important stresses follow from those already tabulated. When $\lambda = 0\cdot5$, the values of $\widehat{\theta\theta}$ at the edge of the hole are as shown in Table XXXIII. The stress being

TABLE XXXIII.—Values of $\frac{b}{\text{P}}\,\widehat{\theta\theta}$ round the hole when the strip is loaded as in fig. 8.

θ	$\frac{b}{\text{P}}\widehat{\theta\theta}$	θ	$\frac{b}{\text{P}}\widehat{\theta\theta}$
0°	− 6·61	—	—
15°	− 8·18	− 15°	− 2·58
30°	− 7·06	− 30°	2·06
45°	− 4·80	− 45°	5·18
60°	− 2·41	− 60°	6·05
75°	− 0·91	− 75°	3·95
90°	0	− 90°	0

* *Cf.* " B," pp. 110 *et seq.*; also FILON, ' Phil. Trans.' A., vol. 334, pp. 63 *et seq.* (1903).

odd in y, the values for $90° < \theta < 270°$ may be written down by inspection. The distribution round the hole is shown in fig. 9.

The effect of the bending-moment is clearly predominant. The numerically largest value of $\widehat{\theta\theta}\, b/\mathrm{P}$, namely $-8 \cdot 18$ at $\theta = 15°$, contains a contribution of $-5 \cdot 38$ from the bending-moment solution. This means that although the stresses round the hole are large, they are relatively of less importance than in the solution for "bending-moment with shear." For the large bending-moment across the middle section now

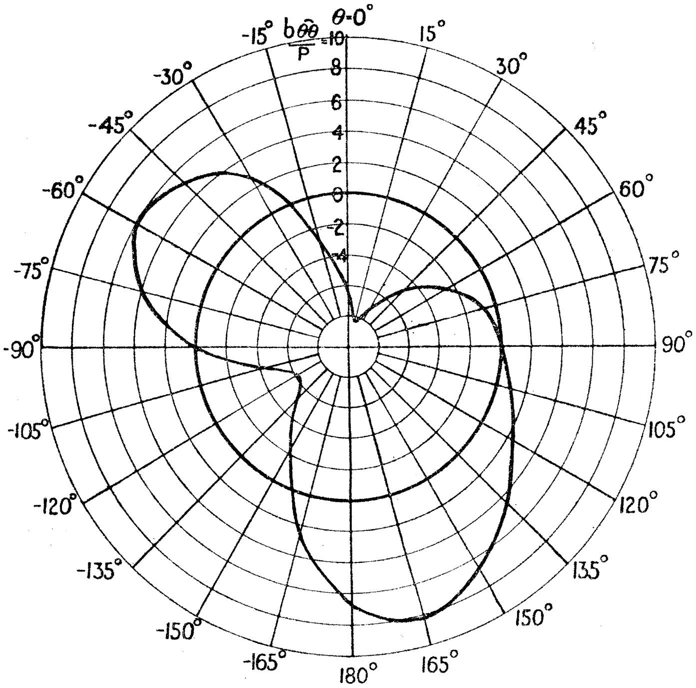

Fig. 9.—*Solution under the Conditions of fig.* 8.—Values of $\widehat{\theta\theta}\, b/\mathrm{P}$ at the edge of the hole.

gives a tension of magnitude $5 \cdot 68\, b/\mathrm{P}$ at the edge of the strip. The greatest tension at the rim of the hole is less than one and a half times this value, and is actually not so large as the tensions that will appear in the edge of the strip at some distance to the left of the hole.

§ 10. Concluding Remarks.

We have now developed the theory to a point at which it is possible to obtain the stress distribution due to any set of forces applied to the perforated strip by means of a straightforward calculation. The functions entering into the solution have been discussed and tabulated and tables of all the necessary coefficients have been given. Among the many special cases of interest we have, however, chosen for a full discussion here only those that hold a fundamental place in the theory. The three solutions for isolated forces discussed in § 3, and the three solutions considered in § 7 are all of this fundamental nature since any of them may be required in the building up of further solutions.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Among other special forms of solution which will be of interest are the group corresponding to those worked out by BICKLEY* for an infinite plate, and taken by him as representing the action of a rivet. One such solution for the strip was outlined in a previous paper† and has been worked out more fully since that time. But the meaning of the results did not appear to us to be quite clear, and this solution has been held back until it can be compared with other solutions and, if possible, with experimental results.

Our analysis can, with slight changes of detail, also be applied to other problems, notably that of the motion of a cylinder in a channel of viscous liquid, a problem considered by BAIRSTOW‡ by another method, but solved only for a channel of breadth large compared with that of the cylinder. This problem again is identical, as far as the analysis is concerned, with that of the bending of a perforated strip. To these problems we hope to return in a later communication.§

§ 11. LIST OF SYMBOLS EMPLOYED, WITH THE PAGES ON WHICH THEY ARE DEFINED.

[Symbols used once only, and defined when used, and dummy symbols, such as the variables in a definite integral or a summation, are omitted from the following list.]

	Page
x, y, r, θ	156
ξ, η, ρ, λ	157
a, b	156
χ	157
$\widehat{xx}, \widehat{xy}, \widehat{yy}, \widehat{rr}, \widehat{r\theta}, \widehat{\theta\theta}$	157
χ', χ'', χ'''	157
σ	159
$\chi_\alpha, \chi_\beta, \chi_\gamma$	159, 161, 165
$\Phi_\alpha, \Phi_\beta, \Phi_\gamma$	159, 161, 165
s, c, S, C, Σ	159
Σ'	161
${}^\alpha a_n, {}^\alpha b_n, {}^\beta a_n, {}^\beta b_n, {}^\gamma a_n, {}^\gamma b_n$	160, 162, 166
${}^\alpha a'_n, {}^\alpha b'_n, {}^\beta a'_n, {}^\beta b'_n, {}^\gamma a'_n, {}^\gamma b'_n$	160, 164, 166
${}^\alpha p_n, {}^\alpha r_n, {}^\beta p_n, {}^\beta r_n, {}^\gamma p_n, {}^\gamma r_n$	161, 164, 166
$I_s, J_s, I'_s, J'_s, (s > 2)$	160, 162
$I'_0, I'_1, I'_2, J'_1, J'_2$	163
S_n, T_n, U_n, V_n	176, 181, 188, 195
χ_{2r}, χ_{2r+1}	176, 181, 184, 188, 190, 195, 197

* 'Phil. Trans.,' A, vol. 227, pp. 383–415 (1928).

† *loc. cit. ante.*, p. 158.

‡ 'Proc. Roy. Soc.,' A, vol. 100, pp. 394–413 (1922).

§ A solution for the rotation of a cylinder in a channel is given by HOWLAND and KNIGHT in a paper about to be published :—'Proc. Camb. Phil. Soc.' (1933).

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES